On a transform of an acyclic complex of length 3

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Abstract

Let $(R\,,\mathfrak{m})$ be a Cohen-Macaulay local ring and Q a parameter ideal of $R\,$. Suppose that an acyclic complex $F_\bullet$ of length 3 which is an R-free resolution of an ideal $\mathfrak{a}$ of R is given. In this paper, we describe a concrete procedure to get an acyclic complex ${}^*F_\bullet$ of length 3 that becomes an R-free resolution of $\mathfrak{a} :_R Q\,$. As an application, we compute the symbolic powers of ideals generated by maximal minors of certain 2×3 matrices.

1 Introduction

Let $(R, \mathfrak{m})$ be a 3-dimensional Cohen-Macaulay local ring and x_1, x_2, x_3 an sop for $R\,$. We put $Q = (x_1, x_2, x_3)R\,$. Suppose that an acyclic complex

$$0 \longrightarrow F_3 \xrightarrow{\varphi_3} F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 = R$$

of finitely generated free R-modules such that $\operatorname{Im}\varphi_3 \subseteq QF_2$ is given. In this paper, we describe an operation to get an acyclic complex

$$0 \longrightarrow {}^*F_3 \xrightarrow{{}^*\varphi_3} {}^*F_2 \xrightarrow{{}^*\varphi_2} {}^*F_1 \xrightarrow{{}^*\varphi_1} {}^*F_0 = R$$

such that $\operatorname{Im}{}^*\varphi_1 = \operatorname{Im}\varphi_1 :_R Q$ and $\operatorname{Im}{}^*\varphi_3 \subseteq \mathfrak{m}\,{}^*F_2\,$, which is called the $*$-transform of $F_\bullet$ with respect to $x_1, x_2, x_3\,$. As we give a practical condition for ${}^*F_3$ to be vanished, it is possible to consider when the depth of $R/(\operatorname{Im}\varphi_1 :_R Q)$ is positive.

If R is a regular local ring and $\mathfrak{a}$ is an ideal of $R\,$, taking a regular sop and the minimal free resolution of $\mathfrak{a}$ as x_1, x_2, x_3 and $F_\bullet\,$, respectively, we get an acyclic complex ${}^*F_\bullet\,$, which is a free resolution of $\mathfrak{a} :_R \mathfrak{m}\,$. Here, let us notice that we can again take the $*$-transform of ${}^*F_\bullet$ with respect to a regular sop since $\operatorname{Im}{}^*\varphi_3 \subseteq \mathfrak{m}\,{}^*F_2\,$, and a free resolution of $\mathfrak{a} :_R \mathfrak{m}^2$ is induced. Repeating this operation, for any $k > 0\,$, we get a free resolution of $\mathfrak{a} :_R \mathfrak{m}^k\,$. Moreover, in the case where $\dim R/\mathfrak{a} > 0\,$, it is possible to find the minimal integer $k > 0$ such that $\operatorname{depth} R/(\mathfrak{a} :_R \mathfrak{m}^k) > 0\,$.

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As an application of the transformation stated above, we can compute the symbolic powers of an ideal I generated by the maximal minors of the matrix

$$\begin{pmatrix} x^\alpha & y^\beta & z^\gamma \\ y^{\beta'} & z^{\gamma'} & x^{\alpha'} \end{pmatrix},$$

where x, y, z is an sop for R and $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ are positive integers. As is well known, R/I is a Cohen-Macaulay local ring with $\dim R/I = 1$. The n-th symbolic power of I is defined by

$$I^{(n)} = \bigcap_{\mathfrak{p} \in \mathrm{Min}_R(R/I)} I^n R_{\mathfrak{p}} \cap R,$$

and so, if J is an $\mathfrak{m}$-primary ideal such that $\operatorname{depth} R/(I^n :_R J) > 0$, we have $I^{(n)} = I^n :_R J$. Because I is generated by a d-sequence, it is possible to describe a minimal free resolution of I^n, and we can take its $*$-transform with respect to $x^{\alpha''}, y^{\beta''}, z^{\gamma''}$, where $\alpha'' = \min\{\alpha, \alpha'\}$, $\beta'' = \min\{\beta, \beta'\}$ and $\gamma'' = \min\{\gamma, \gamma'\}$. If the length of the resulting acyclic complex is still 3, we have $\operatorname{depth} R/(I^n :_R (x^{\alpha''}, y^{\beta''}, z^{\gamma''})) = 0$, and it means $I^n :_R (x^{\alpha''}, y^{\beta''}, z^{\gamma''}) \subsetneq I^{(n)}$. Then, we take once more the $*$-transform with respect to suitable powers of x, y, z. By repeating such operation several times, we can reach $I^{(n)}$. In Section 4, we carry it out for $n = 2, 3$. Our method is completely different from that used in [2] and [5], in which $I^{(2)}$ and $I^{(3)}$ are computed in the case where R is a regular local ring with the maximal ideal $\mathfrak{m} = (x, y, z)R$ and I is a prime ideal.

In the last section, assuming that R is a regular local ring and x, y, z is a regular sop for R, we compute the length of $I^{(n)}/I^n$ for all $n \geq 1$ in the case where I is generated by the maximal minors of the matrix

$$\begin{pmatrix} x & y & z \\ y & z & x^2 \end{pmatrix}.$$

Starting from a minimal free resolution of I^n, we repeat the operation to take the $*$-transform with respect to x, y, z successively. Then we get a free resolution of $I^n :_R \mathfrak{m}^k$ for any $k \geq 1$, and it follows that $I^{(n)} = I^n :_R \mathfrak{m}^q$, where q is the largest integer with $q \leq n/2$. As our method enables us to compute the length of $(I^n :_R \mathfrak{m}^k)/(I^n :_R \mathfrak{m}^{k-1})$ for any $k \geq 1$, we get to know the length of $I^{(n)}/I^n$ exactly. As a consequence of our result, we see that the ϵ-multiplicity of I, which is an invariant defined by

$$\epsilon(I) := \lim_{n \to \infty} \frac{3!}{n^3} \cdot \ell_R(\, I^{(n)}/I^n \,)$$

(cf. [1], [6]), coincides with $1/2$.

Throughout this paper, R is a 3-dimensional Cohen-Macaulay local ring with the maximal ideal $\mathfrak{m}$. If an R-module is a direct sum of finite number of R-modules, its elements are denoted by column vectors. In particular, if an R-module F is a direct sum of two R-modules, say $F = G \oplus H$, the elements in F of the form

$$\begin{pmatrix} g \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ h \end{pmatrix} \qquad (g \in G,\ h \in H)$$

are denoted by $[g]$ and $\langle h \rangle$, respectively. Moreover, if V is a subset of G, then the family $\{\, [g] \,\}_{g \in V}$ is denoted by $[V]$. Similarly $\langle W \rangle$ is defined for a subset W of H. If T is a subset of an R-module, we denote by $R{\cdot}T$ the R-submodule generated by T.

2 Preliminaries

In this section, we summarize preliminary results. Although they might be well known, we give the proofs for completeness. Let us begin with the following lemma.

Lemma 2.1 *Let $G_\bullet$ and $F_\bullet$ be acyclic complexes, whose boundary maps are denoted by $\partial_\bullet$ and $\varphi_\bullet$, respectively. Suppose that a chain map $\sigma_\bullet : G_\bullet \longrightarrow F_\bullet$ is given and $\sigma_0^{-1}(\operatorname{Im}\varphi_1) = \operatorname{Im}\partial_1$ holds. Then the mapping cone $\operatorname{Con}(\sigma_\bullet)$ is acyclic. Hence, if $G_\bullet$ and $F_\bullet$ are complexes of finitely generated free R-modules, then $\operatorname{Con}(\sigma_\bullet)$ gives an R-free resolution of $\operatorname{Im}\varphi_1 + \operatorname{Im}\sigma_0$.*

Proof. We put $C_\bullet = \operatorname{Con}(\sigma_\bullet)$ and denote its boundary map by $d_\bullet$. Then

$$C_i = F_i \oplus G_{i-1}\ , \qquad d_i = \begin{pmatrix} \varphi_i & (-1)^{i-1}\sigma_{i-1} \\ 0 & \partial_{i-1} \end{pmatrix}$$

for any $i \geq 2$ and

$$C_1 = F_1 \oplus G_0\ , \qquad C_0 = F_0\ , \qquad d_1 = \begin{pmatrix} \varphi_1 & \sigma_0 \end{pmatrix}.$$

Let us take any

$$\begin{pmatrix} x_1 \\ y_0 \end{pmatrix} \in \operatorname{Ker} d_1\,.$$

Then $\varphi_1(x_1) + \sigma_0(y_0) = 0$, so $y_0 \in \sigma_0^{-1}(\operatorname{Im}\varphi_1) = \operatorname{Im}\partial_1$. Hence there exists $y_1 \in G_1$ such that $y_0 = \partial_1(y_1)$. We have $\varphi_1(\partial_1(y_1)) = \sigma_0(\partial_1(y_1)) = \sigma_0(y_0) = -\varphi_1(x_1)$, and so, $\varphi_1(x_1 + \sigma_1(y_1)) = 0$. This means that there exists $x_2 \in F_2$ such that $x_1 + \sigma_1(y_1) = \varphi_2(x_2)$ as $\mathrm{H}_1(F_\bullet) = 0$. Then

$$\begin{pmatrix} x_1 \\ y_0 \end{pmatrix} = \begin{pmatrix} \varphi_2(x_2) - \sigma_1(y_1) \\ \partial_1(y_1) \end{pmatrix} = d_2\begin{pmatrix} x_2 \\ y_1 \end{pmatrix} \in \operatorname{Im} d_2\,.$$

Therefore we get $\mathrm{H}_1(C_\bullet) = 0$. On the other hand, the exact sequence

$$0 \longrightarrow G_\bullet(-1) \longrightarrow C_\bullet \longrightarrow F_\bullet \longrightarrow 0$$

of complexes induces a long exact sequence

$$\cdots \longrightarrow \mathrm{H}_{i-1}(G_\bullet) \longrightarrow \mathrm{H}_i(C_\bullet) \longrightarrow \mathrm{H}_i(F_\bullet) \longrightarrow \cdots$$

of homology modules. If $i \geq 2$, $\mathrm{H}_{i-1}(G_\bullet) = \mathrm{H}_i(F_\bullet) = 0$, and so, $\mathrm{H}_i(C_\bullet) = 0$. Thus we have seen that $C_\bullet$ is an acyclic complex.

Lemma 2.2 *Let $C_{\bullet\bullet}$ be a double complex such that $C_{pq} = 0$ unless $0 \leq p, q \leq 3$. For any $p, q \in \mathbb{Z}$, we denote the boundary maps $C_{pq} \longrightarrow C_{p-1,q}$ and $C_{pq} \longrightarrow C_{p,q-1}$ by d'_{pq} and d''_{pq}, respectively. If $C_{p\bullet}$ and $C_{\bullet q}$ are acyclic for any $0 \leq p, q \leq 3$, we have the following assertions on the total complex $T_\bullet$, whose boundary map is denoted by $d_\bullet$.*

(1) *Suppose that* $\xi_0 \in C_{0,3}$ *and* $\xi_1 \in C_{1,2}$ *such that* $d''_{0,3}(\xi_0) + d'_{1,2}(\xi_1) = 0$ *are given. Then there exist* $\xi_2 \in C_{2,1}$ *and* $\xi_3 \in C_{3,0}$ *such that* ${}^t\!\begin{pmatrix}\xi_0 & \xi_1 & \xi_2 & \xi_3\end{pmatrix} \in \operatorname{Ker} d_3 \subseteq T_3 = C_{0,3} \oplus C_{1,2} \oplus C_{2,1} \oplus C_{3,0}$.

(2) *Let* ${}^t\!\begin{pmatrix}\xi_0 & \xi_1 & \xi_2 & \xi_3\end{pmatrix} \in \operatorname{Ker} d_3 \subseteq T_3 = C_{0,3} \oplus C_{1,2} \oplus C_{2,1} \oplus C_{3,0}$ *and let* $\xi_3 \in \operatorname{Im} d''_{3,1}$. *Then* ${}^t\!\begin{pmatrix}\xi_0 & \xi_1 & \xi_2 & \xi_3\end{pmatrix} \in \operatorname{Im} d_4$. *In particular,* $\xi_0 \in \operatorname{Im} d'_{1,3}$.

Proof. (1) Because $d'_{1,1}(d''_{1,2}(\xi_1)) = d''_{0,2}(d'_{1,2}(\xi_1)) = d''_{0,2}(-d''_{0,3}(\xi_0)) = 0$, it follows that $d''_{1,2}(\xi_1) \in \operatorname{Ker} d'_{1,1} = \operatorname{Im} d'_{2,1}$. So, we can write $d''_{1,2}(\xi_1) = d'_{2,1}(\xi_2)$, where $\xi_2 \in C_{2,1}$. Then $d'_{2,0}(d''_{2,1}(\xi_2)) = d''_{1,1}(d'_{2,1}(\xi_2)) = d''_{1,1}(d''_{1,2}(\xi_1)) = 0$, and so $d''_{2,1}(\xi_2) \in \operatorname{Ker} d'_{2,0} = \operatorname{Im} d'_{3,0}$. Hence there exists $\xi_3 \in C_{3,0}$ such that $d''_{2,1}(\xi_2) = -d'_{3,0}(\xi_3)$. Now we have

$$d_3 \begin{pmatrix}\xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3\end{pmatrix} = \begin{pmatrix} d''_{0,3}(\xi_0) + d'_{1,2}(\xi_1) \\ -d''_{1,2}(\xi_1) + d'_{2,1}(\xi_2) \\ d''_{2,1}(\xi_2) + d'_{3,0}(\xi_3)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} ,$$

and so we get the required assertion.

(2) Because ${}^t\!\begin{pmatrix}\xi_0 & \xi_1 & \xi_2 & \xi_3\end{pmatrix} \in \operatorname{Ker} d_3$, we have (i) $d''_{0,3}(\xi_0) + d'_{1,2}(\xi_1) = 0$, (ii) $-d''_{1,2}(\xi_1) + d'_{2,1}(\xi_2) = 0$ and (iii) $d''_{2,1}(\xi_2) + d'_{3,0}(\xi_3) = 0$. On the other hand, as $\xi_3 \in \operatorname{Im} d''_{3,1}$, there exists $\eta_3 \in C_{3,1}$ such that $\xi_3 = -d''_{3,1}(\eta_3)$. Then, as $d'_{3,0}(\xi_3) = -d'_{3,0}(d''_{3,1}(\eta_3)) = -d''_{2,1}(d'_{3,1}(\eta_3))$, by (iii) we get $\xi_2 - d'_{3,1}(\eta_3) \in \operatorname{Ker} d''_{2,1} = \operatorname{Im} d''_{2,2}$. So, we can write $\xi_2 - d'_{3,1}(\eta_3) = d''_{2,2}(\eta_2)$, where $\eta_2 \in C_{2,2}$. Then $d'_{2,1}(\xi_2) = d'_{2,1}(d'_{3,1}(\eta_3) + d''_{2,2}(\eta_2)) = d'_{2,1}(d''_{2,2}(\eta_2)) = d''_{1,2}(d'_{2,2}(\eta_2))$, and so by (ii) we get $\xi_1 - d'_{2,2}(\eta_2) \in \operatorname{Ker} d''_{1,2} = \operatorname{Im} d''_{1,3}$. Hence there exists $\eta_1 \in C_{1,3}$ such that $\xi_1 - d'_{2,2}(\eta_2) = -d''_{1,3}(\eta_1)$. This means $d'_{1,2}(\xi_1) = d'_{1,2}(d'_{2,2}(\eta_2) - d''_{1,3}(\eta_1)) = -d'_{1,2}(d''_{1,3}(\eta_1)) = -d''_{0,3}(d'_{1,3}(\eta_1))$, and so by (i) we get $\xi_0 - d'_{1,3}(\eta_1) \in \operatorname{Ker} d''_{0,3}$. However, as $C_{0,\bullet}$ is acyclic and $C_{0,4} = 0$, $d''_{0,3}$ is injective. Hence $\xi_0 = d'_{1,3}(\eta_1)$. Now we have

$$d_4 \begin{pmatrix}\eta_1 \\ \eta_2 \\ \eta_3\end{pmatrix} = \begin{pmatrix} d'_{1,3}(\eta_1) \\ -d''_{1,3}(\eta_1) + d'_{2,2}(\eta_2) \\ d''_{2,2}(\eta_2) + d'_{3,1}(\eta_3) \\ -d''_{3,1}(\eta_3) \end{pmatrix} = \begin{pmatrix}\xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3\end{pmatrix} ,$$

and the proof is completed.

Lemma 2.3 *Suppose that*

$$0 \longrightarrow F \xrightarrow{\varphi} G \xrightarrow{\psi} H \xrightarrow{\rho} L$$

is an exact sequence of R*-modules. Then the following assertions hold.*

(1) *If there exists a homomorphism* $\phi : G \longrightarrow F$ *of* R*-modules such that* $\phi \circ \varphi = \mathrm{id}_F$ *, then*

$$0 \longrightarrow {}^*G \xrightarrow{{}^*\psi} H \xrightarrow{\rho} L$$

is exact, where ${}^*G = \operatorname{Ker} \phi$ *and* ${}^*\psi$ *is the restriction of* ψ *to* ${}^*G$.

(2) *If $F = {'F} \oplus {^*F}$, $G = {'G} \oplus {^*G}$, $\varphi({'F}) = {'G}$ and $\varphi({^*F}) \subseteq {^*G}$, then*

$$0 \longrightarrow {^*F} \xrightarrow{^*\varphi} {^*G} \xrightarrow{^*\psi} H \xrightarrow{\rho} L$$

is exact, where ${^*\varphi}$ and ${^*\psi}$ are the restrictions of φ and ψ to ${^*F}$ and ${^*G}$, respectively.

Proof. (1) First, we take any $u \in \operatorname{Ker} \rho$. As $\operatorname{Im} \psi = \operatorname{Ker} \rho$, there exists $v \in G$ such that $\psi(v) = u$. Then, setting ${^*v} = v - \varphi(\phi(v))$, we have ${^*v} \in {^*G}$ and $\psi({^*v}) = \psi(v) = u$, and so $u \in \psi({^*G})$. Hence $\operatorname{Ker} \rho = \operatorname{Im} {^*\psi}$.

Next, we take any ${^*v} \in {^*G}$ such that $\psi({^*v}) = 0$. As $\operatorname{Im} \varphi = \operatorname{Ker} \psi$, there exists $w \in F$ such that $\varphi(w) = {^*v}$. Then $w = \phi(\varphi(w)) = \phi({^*v}) = 0$, and so ${^*v} = \varphi(0) = 0$. Hence ${^*\varphi}$ is injective.

(2) First, we take any $u \in \operatorname{Ker} \rho$. As $\operatorname{Im} \psi = \operatorname{Ker} \rho$, there exists $v \in G$ such that $\psi(v) = u$. We write $v = {'v} + {^*v}$, where ${'v} \in {'G}$ and ${^*v} \in {^*G}$, and choose ${'w} \in {'F}$ so that $\varphi({'w}) = {'v}$. Then we have $u = \psi(\varphi({'w}) + {^*v}) = \psi({^*v}) \in \psi({^*G})$. Hence $\operatorname{Im} {^*\psi} = \operatorname{Ker} \rho$.

Next, we take any ${^*v} \in {^*G}$ such that $\psi({^*v}) = 0$. As $\operatorname{Im} \varphi = \operatorname{Ker} \psi$, there exists $w \in F$ such that $\varphi(w) = {^*v}$. We write $w = {'w} + {^*w}$, where ${'w} \in {'F}$ and ${^*w} \in {^*F}$. Then ${^*v} - \varphi({^*w}) = \varphi({'w}) \in {'G} \cap {^*G} = 0$. This means ${^*v} = \varphi({^*w}) \in \varphi({^*G})$. Hence $\operatorname{Im} {^*\varphi} = \operatorname{Ker} {^*\psi}$, and the proof is completed.

3 $*$-transform

Let R be a 3-dimensional Cohen-Macaulay local ring with the maximal ideal $\mathfrak{m}$ and

$$0 \longrightarrow F_3 \xrightarrow{\varphi_3} F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 = R$$

be an acyclic complex of finitely generated free R-modules such that $\operatorname{Im} \varphi_3 \subseteq QF_2$, where $Q = (x_1, x_2, x_3)$ is a parameter ideal of R . We put $\mathfrak{a} = \operatorname{Im} \varphi_1$, which is an ideal of R . In this section, transforming $F_\bullet$ suitably, we aim to construct an acyclic complex

$$0 \longrightarrow {^*F_3} \xrightarrow{^*\varphi_3} {^*F_2} \xrightarrow{^*\varphi_2} {^*F_1} \xrightarrow{^*\varphi_1} {^*F_0} = R$$

of finitely generated free R-modules such that $\operatorname{Im} {^*\varphi_3} \subseteq \mathfrak{m}{^*F_2}$ and $\operatorname{Im} {^*\varphi_1} = \mathfrak{a} :_R Q$. Let us call ${^*F_\bullet}$ the $*$-transform of $F_\bullet$ with respect to x_1, x_2, x_3 .

In this operation, we use the Koszul complex $K_\bullet = K_\bullet(x_1, x_2, x_3)$. Let e_1, e_2, e_3 be an R-free basis of K_1 and set $\check{e}_1 = e_2 \wedge e_3$, $\check{e}_2 = e_1 \wedge e_3$, $\check{e}_3 = e_1 \wedge e_2$. Then $\check{e}_1, \check{e}_2, \check{e}_3$ is an R-free basis of K_2 . Furthermore $e_1 \wedge e_2 \wedge e_3$ is an R-free basis of K_3 . The boundary maps of $K_\bullet$ which we denote by $\partial_\bullet$ satisfy

$$\begin{aligned} \partial_1(e_i) &= x_i \quad \text{for any } i = 1, 2, 3 , \\ \partial_2(e_i \wedge e_j) &= x_i e_j - x_j e_i \quad \text{if } 1 \le i < j \le 3 , \\ \partial_3(e_1 \wedge e_2 \wedge e_3) &= x_1 \check{e}_1 - x_2 \check{e}_2 + x_3 \check{e}_3 . \end{aligned}$$

As x_1, x_2, x_3 is an R-regular sequence, $K_\bullet$ gives an R-free resolution of R/Q . Hence, for any R-module M , We have the following commutative diagram;

$$\begin{array}{ccccccccc} \operatorname{Hom}_R(K_2, M) & \xrightarrow{\operatorname{Hom}_R(\partial_3, M)} & \operatorname{Hom}_R(K_3, M) & \longrightarrow & \operatorname{Ext}^3_R(R/Q, M) & \longrightarrow & 0 & (\mathrm{ex}) \\ \downarrow \cong & & \downarrow \cong & & & & & \\ M^{\oplus 3} & \xrightarrow{(x_1\ -x_2\ x_3)} & M & \longrightarrow & M/QM & \longrightarrow & 0 & (\mathrm{ex}) , \end{array}$$

which implies $\mathrm{Ext}_R^3(R/Q\,,M) \cong M/QM$. Using this fact, we show the next result.

Theorem 3.1 $(\mathfrak{a} :_R Q)/\mathfrak{a} \cong F_3/QF_3$.

Proof. As is well known, we have

$$(\mathfrak{a} :_R Q)/\mathfrak{a} \cong \mathrm{Hom}_R(R/Q\,, R/\mathfrak{a})\,.$$

Applying $\mathrm{Hom}_R(R/Q, \bullet\,)$ to the exact sequence $0 \longrightarrow \mathfrak{a} \longrightarrow R \longrightarrow R/\mathfrak{a} \longrightarrow 0$, we get

$$\mathrm{Hom}_R(R/Q\,, R/\mathfrak{a}) \cong \mathrm{Ext}_R^1(R/Q\,, \mathfrak{a})$$

as $\mathrm{Hom}_R(R/Q\,, R) = \mathrm{Ext}_R^1(R/Q\,, R) = 0$. We put $L = \mathrm{Ker}\,\varphi_1 = \mathrm{Im}\,\varphi_2$ and consider the exact sequences

$$0 \longrightarrow L \longrightarrow F_1 \xrightarrow{\varphi_1} \mathfrak{a} \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow F_3 \xrightarrow{\varphi_3} F_2 \longrightarrow L \longrightarrow 0\,.$$

Because $\mathrm{Ext}_R^i(R/Q\,, F_j) = 0$ for any $0 \le i \le 2$ and $0 \le j \le 3$, we get

$$\mathrm{Ext}_R^1(R/Q\,, \mathfrak{a}) \cong \mathrm{Ext}_R^2(R/Q\,, L)$$

and the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ext}_R^2(R/Q\,, L) & \longrightarrow & \mathrm{Ext}_R^3(R/Q\,, F_3) & \xrightarrow{\tilde{\varphi}_3} & \mathrm{Ext}_R^3(R/Q\,, F_2) & \text{(ex)} \\
 & & & & \downarrow \cong & & \downarrow \cong & \\
 & & & & F_3/QF_3 & \xrightarrow{\overline{\varphi}_3} & F_2/QF_2 & \text{(ex)}\,,
\end{array}$$

where $\tilde{\varphi}_3$ and $\overline{\varphi}_3$ denote the maps induced from φ_3 . Let us notice $\overline{\varphi}_3 = 0$ as $\mathrm{Im}\,\varphi_3 \subseteq QF_2$. Hence

$$\mathrm{Ext}_R^2(R/Q\,, L) \cong F_3/QF_3\,,$$

and so the required isomorphism follows.

Let us fix an R-free basis of F_3 , say $\{\, w_\lambda \,\}_{\lambda \in \Lambda}$. Let $\{\, v_{(i,\lambda)} \,\}_{1 \le i \le 3\,,\, \lambda \in \Lambda}$ be a family of elements in F_2 such that

$$\varphi_3(w_\lambda) = \sum_{i=1}^{3} x_i \cdot v_{(i,\lambda)}$$

for any $\lambda \in \Lambda$. We put $\tilde{\Lambda} = \{\, 1\,, 2\,, 3\,\} \times \Lambda$. The next result is the essential part of the process to get ${}^*F_\bullet$.

Theorem 3.2 *There exists a chain map* $\sigma_\bullet : K_\bullet \otimes_R F_3 \longrightarrow F_\bullet$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_3 \otimes_R F_3 & \xrightarrow{\partial_3 \otimes \mathrm{id}} & K_2 \otimes_R F_3 & \xrightarrow{\partial_2 \otimes \mathrm{id}} & K_1 \otimes_R F_3 & \xrightarrow{\partial_1 \otimes \mathrm{id}} & K_0 \otimes_R F_3 \\
 & & \downarrow \sigma_3 & & \downarrow \sigma_2 & & \downarrow \sigma_1 & & \downarrow \sigma_0 \\
0 & \longrightarrow & F_3 & \xrightarrow{\varphi_3} & F_2 & \xrightarrow{\varphi_2} & F_1 & \xrightarrow{\varphi_1} & F_0
\end{array}$$

satisfying the following conditions.

(1) $\sigma_0^{-1}(\mathrm{Im}\,\varphi_1) = \mathrm{Im}\,(\partial_1 \otimes \mathrm{id}_{F_3})$.

(2) $\mathrm{Im}\,\sigma_0 + \mathrm{Im}\,\varphi_1 = \mathfrak{a} :_R Q$.

(3) $\sigma_2(\check{e}_i \otimes w_\lambda) = (-1)^i \cdot v_{(i,\lambda)}$ *for any* $(i, \lambda) \in \tilde{\Lambda}$.

(4) $\sigma_3((e_1 \wedge e_2 \wedge e_3) \otimes w_\lambda) = -w_\lambda$ *for any* $\lambda \in \Lambda$.

Proof. Let us consider the double complex $K_\bullet \otimes_R F_\bullet$.

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & K_i \otimes_R F_j & \xrightarrow{\partial_i \otimes \mathrm{id}_{F_j}} & K_{i-1} \otimes_R F_j & \longrightarrow & \cdots \\
 & & \big\downarrow{\scriptstyle \mathrm{id}_{K_i} \otimes \varphi_j} & & \big\downarrow{\scriptstyle \mathrm{id}_{K_{i-1}} \otimes \varphi_j} & & \\
\cdots & \longrightarrow & K_i \otimes_R F_{j-1} & \xrightarrow{\partial_i \otimes \mathrm{id}_{F_{j-1}}} & K_{i-1} \otimes_R F_{j-1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & &
\end{array}$$

We can take it as $C_{\bullet\bullet}$ in 2.2. Let $T_\bullet$ be the total complex and $d_\bullet$ be its boundary map. For any $\lambda \in \Lambda$, we set $\xi_0(\lambda) = 1 \otimes w_\lambda \in K_0 \otimes F_3$ and $\xi_1(\lambda) = -\sum_{i=1}^{3} e_i \otimes v_{(i,\lambda)} \in K_1 \otimes_R F_2$. Then we have

$$(\mathrm{id}_{K_0} \otimes \varphi_3)(\xi_0(\lambda)) + (\partial_1 \otimes \mathrm{id}_{F_2})(\xi_1(\lambda)) = 0\,,$$

and so by 2.2 there exist $\xi_2(\lambda) \in K_2 \otimes_R F_1$ and $\xi_3(\lambda) \in K_3 \otimes_R F_0$ such that

$${}^t(\,\xi_0(\lambda)\ \ \xi_1(\lambda)\ \ \xi_2(\lambda)\ \ \xi_3(\lambda)\,) \in \mathrm{Ker}\,(T_3 \xrightarrow{d_3} T_2)\,.$$

When this is the case, the following equalities hold;

$$\begin{array}{rl}
\text{(i)} & -\,(\mathrm{id}_{K_1} \otimes \varphi_2)(\xi_1(\lambda)) + (\partial_2 \otimes \mathrm{id}_{F_1})(\xi_2(\lambda)) = 0\,, \\
\text{(ii)} & (\mathrm{id}_{K_2} \otimes \varphi_1)(\xi_2(\lambda)) + (\partial_3 \otimes \mathrm{id}_{F_0})(\xi_3(\lambda)) = 0\,.
\end{array}$$

Here we write

$$\xi_2(\lambda) = -\sum_{i=1}^{3} \check{e}_i \otimes u_{(i,\lambda)} \quad \text{and} \quad \xi_3(\lambda) = (e_1 \wedge e_2 \wedge e_3) \otimes d_\lambda\,,$$

where $u_{(i,\lambda)} \in F_1$ and $d_\lambda \in F_0 = R$. Then by (i) we get

$$\text{(iii)} \quad \begin{cases}
\varphi_2(v_{(1,\lambda)}) = -x_2 \cdot u_{(3,\lambda)} - x_3 \cdot u_{(2,\lambda)} \\
\varphi_2(v_{(2,\lambda)}) = x_1 \cdot u_{(3,\lambda)} - x_3 \cdot u_{(1,\lambda)} \\
\varphi_2(v_{(3,\lambda)}) = x_1 \cdot u_{(2,\lambda)} + x_2 \cdot u_{(1,\lambda)}
\end{cases}$$

for any $\lambda \in \Lambda$. Moreover, the equality (ii) implies

$$\text{(iv)} \quad x_i \cdot d_\lambda = (-1)^{i-1} \cdot \varphi_1(u_{(i,\lambda)})$$

for any $(i,\lambda) \in \tilde{\Lambda}$. Now we define $\sigma_0 : K_0 \otimes_R F_3 \longrightarrow F_0$ and $\sigma_1 : K_1 \otimes_R F_3 \longrightarrow F_1$ by setting $\sigma_0(1 \otimes w_\lambda) = d_\lambda$ for any $\lambda \in \Lambda$ and $\sigma_1(e_i \otimes w_\lambda) = (-1)^{i-1} \cdot u_{(i,\lambda)}$ for any $(i,\lambda) \in \tilde{\Lambda}$. The maps $\sigma_2 : K_2 \otimes_R F_3 \longrightarrow F_2$ and $\sigma_3 : K_3 \otimes_R F_3 \longrightarrow F_3$ are defined as in (3) and (4). Then, by (iii) and (iv), we see that $\sigma_\bullet : K_\bullet \otimes_R F_3 \longrightarrow F_\bullet$ is a chain map.

Let us prove (1). Because $\mathrm{Im}\,(\partial_1 \otimes \mathrm{id}_{F_3})$ is obviously contained in $\sigma_0^{-1}(\mathrm{Im}\,\varphi_1)$, we have to show the converse inclusion. Take any $\eta_0 \in K_0 \otimes_R F_3$ such that $\sigma_0(\eta_0) \in \mathrm{Im}\,\varphi_1$. We write

$$\eta_0 = \sum_{\lambda \in \Lambda} a_\lambda \otimes w_\lambda = \sum_{\lambda \in \Lambda} a_\lambda \cdot \xi_0(\lambda)\,,$$

where $a_\lambda \in R$ for any $\lambda \in \Lambda$, and set

$$\eta_i = \sum_{\lambda \in \Lambda} a_\lambda \cdot \xi_i(\lambda) \in K_i \otimes_R F_{3-i}$$

for $i = 1, 2, 3$. Then,

$${}^t(\,\eta_0 \;\; \eta_1 \;\; \eta_2 \;\; \eta_3\,) = \sum_{\lambda \in \Lambda} a_\lambda \cdot {}^t(\,\xi_0(\lambda) \;\; \xi_1(\lambda) \;\; \xi_2(\lambda) \;\; \xi_3(\lambda)\,) \in \mathrm{Ker}\, d_3\,.$$

Furthermore, we have

$$\begin{aligned} \eta_3 &= \sum_{\lambda \in \Lambda} a_\lambda \cdot ((e_1 \wedge e_2 \wedge e_3) \otimes d_\lambda) \\ &= (e_1 \wedge e_2 \wedge e_3) \otimes \sum_{\lambda \in \Lambda} a_\lambda \cdot \sigma_0(1 \otimes w_\lambda) \\ &= (e_1 \wedge e_2 \wedge e_3) \otimes \sigma_0(\eta_0)\,, \end{aligned}$$

and so $\eta_3 \in \mathrm{Im}\,(\mathrm{id}_{K_3} \otimes \varphi_1)$. Hence $\eta_0 \in \mathrm{Im}\,(\partial_1 \otimes \mathrm{id}_{F_3})$ by (1) of 2.2.

Finally we prove (2). Let us consider the following commutative diagram

$$\begin{array}{ccccccccc} K_1 \otimes_R F_3 & \xrightarrow{\partial_1 \otimes \mathrm{id}_{F_3}} & K_0 \otimes_R F_3 & \longrightarrow & F_3/QF_3 & \longrightarrow & 0 & (\mathrm{ex}) \\ \downarrow \sigma_1 & & \downarrow \sigma_0 & & \downarrow \overline{\sigma}_0 & & & \\ F_1 & \xrightarrow{\varphi_1} & F_0 & \longrightarrow & R/\mathfrak{a} & \longrightarrow & 0 & (\mathrm{ex})\,. \end{array}$$

where $\overline{\sigma}_0$ is the map indeuced from σ_0. Notice that by (iv) we have $d_\lambda \in \mathfrak{a} :_R Q$ for any $\lambda \in \Lambda$, and so $\mathrm{Im}\,\sigma_0 \subseteq \mathfrak{a} :_R Q$. Hence $\mathrm{Im}\,\overline{\sigma}_0 \subseteq (\mathfrak{a} :_R Q)/\mathfrak{a}$. On the other hand, as $\sigma_0^{-1}(\mathrm{Im}\,\varphi_1) = \mathrm{Im}\,(\partial_1 \otimes \mathrm{id}_{F_3})$, we see that $\overline{\sigma}_0$ is injective. Therefore we get $\mathrm{Im}\,\overline{\sigma}_0 = (\mathfrak{a} :_R Q)/\mathfrak{a}$ since $(\mathfrak{a} :_R Q)/\mathfrak{a} \cong F_3/QF_3$ by 3.1 and F_3/QF_3 has a finite length. Thus the assertion (2) follows and the proof is completed.

In the rest, $\sigma_\bullet : K_\bullet \otimes_R F_3 \longrightarrow F_\bullet$ is the chain map constructed in 3.2. Then, by 2.1 the mapping cone $\mathrm{Con}\,(\sigma_\bullet)$ gives an R-free resolution of $\mathfrak{a} :_R Q$, that is,

$$0 \to K_3 \otimes_R F_3 \xrightarrow{\psi_4} \begin{array}{c} K_2 \otimes_R F_3 \\ \oplus \\ F_3 \end{array} \xrightarrow{\psi_3} \begin{array}{c} K_1 \otimes_R F_3 \\ \oplus \\ F_2 \end{array} \xrightarrow{{}'\varphi_2} \begin{array}{c} K_0 \otimes_R F_3 \\ \oplus \\ F_1 \end{array} \xrightarrow{{}^*\varphi_1} F_0 = R\,,$$

is acyclic and $\operatorname{Im} {}^*\varphi_1 = \mathfrak{a} :_R Q$, where

$$\psi_4 = \begin{pmatrix} \partial_3 \otimes \mathrm{id}_{F_3} \\ -\sigma_3 \end{pmatrix}, \ \psi_3 = \begin{pmatrix} \partial_2 \otimes \mathrm{id}_{F_3} & 0 \\ \sigma_2 & \varphi_3 \end{pmatrix}, \ {}'\varphi_2 = \begin{pmatrix} \partial_1 \otimes \mathrm{id}_{F_3} & 0 \\ -\sigma_1 & \varphi_2 \end{pmatrix}, \ {}^*\varphi_1 = (\sigma_0 \ \ \varphi_1) \, .$$

Because $\sigma_3 : K_3 \otimes_R F_3 \longrightarrow F_3$ is an isomorphism, we can define

$$\phi = (\, 0 \quad -\sigma_3^{-1}\,) \ : \ \begin{matrix} K_2 \otimes_R F_3 \\ \oplus \\ F_3 \end{matrix} \ \longrightarrow \ K_3 \otimes_R F_3 \, .$$

Then $\phi \circ \psi_4 = \mathrm{id}_{K_3 \otimes_R F_3}$ and $\operatorname{Ker} \phi = K_2 \otimes_R F_3$. Hence, by (1) of 2.3, we get the acyclic complex

$$0 \longrightarrow {}'F_3 \xrightarrow{{}'\varphi_3} {}'F_2 \xrightarrow{{}'\varphi_2} {}^*F_1 \xrightarrow{{}^*\varphi_1} {}^*F_0 = R \, ,$$

where

$${}'F_3 = K_2 \otimes_R F_3 \, , \quad {}'F_2 = \begin{matrix} K_1 \otimes_R F_3 \\ \oplus \\ F_2 \end{matrix} \, , \quad {}^*F_1 = \begin{matrix} K_0 \otimes_R F_3 \\ \oplus \\ F_1 \end{matrix} \quad \text{and} \quad {}'\varphi_3 = \begin{pmatrix} \partial_2 \otimes \mathrm{id}_{F_3} \\ \sigma_2 \end{pmatrix} .$$

Although $\operatorname{Im} {}'\varphi_3$ may not be contained in $\mathfrak{m}\,{}'F_2$, removing non-minimal components from ${}'F_3$ and ${}'F_2$, we get free R-modules ${}^*F_3$ and ${}^*F_2$ such that

$$0 \longrightarrow {}^*F_3 \xrightarrow{{}^*\varphi_3} {}^*F_2 \xrightarrow{{}^*\varphi_2} {}^*F_1 \xrightarrow{{}^*\varphi_1} {}^*F_0 = R$$

is acyclic and $\operatorname{Im} {}^*\varphi_3 \subseteq \mathfrak{m}\,{}^*F_2$, where ${}^*\varphi_3$ and ${}^*\varphi_2$ are the restrictions of ${}'\varphi_3$ and ${}'\varphi_2$, respectively. In the rest of this section, we describe a concrete procedure to get ${}^*F_3$ and ${}^*F_2$. For that purpose, we use the following notation. As is described in Introduction, for any $\xi \in K_1 \otimes_R F_3$ and $\eta \in F_2$,

$$[\xi] := \begin{pmatrix} \xi \\ 0 \end{pmatrix} \in {}'F_2 \quad \text{and} \quad \langle \eta \rangle := \begin{pmatrix} 0 \\ \eta \end{pmatrix} \in {}'F_2 \, .$$

In particular, for any $(i, \lambda) \in \tilde{\Lambda}$, we denote $[e_i \otimes w_\lambda]$ by $[i, \lambda]$. Moreover, for a subset $U \subseteq F_2$, $\langle U \rangle := \{ \langle u \rangle \}_{u \in U}$.

Now, let us choose a subset ${}'\Lambda$ of $\tilde{\Lambda}$ and a subset U of F_2 so that

$$\{ v_{(i,\lambda)} \}_{(i,\lambda) \in {}'\Lambda} \cup U$$

is an R-free basis of F_2 . We would like to choose ${}'\Lambda$ as big as possible. The following almost obvious fact is useful to find ${}'\Lambda$ and U .

Lemma 3.3 *Let V be an R-free basis of F_2 . If a subset ${}'\Lambda$ of $\tilde{\Lambda}$ and a subset U of V satisfy*

(i) $|\,{}'\Lambda\,| + |\,U\,| \leq |\,V\,|$, *where* $|\cdot|$ *denotes the number of elements of the set, and*

(ii) $V \subseteq R \cdot \{ v_{(i,\lambda)} \}_{(i,\lambda) \in {}'\Lambda} + R \cdot U + \mathfrak{m} F_2$,

then $\{ v_{(i,\lambda)} \}_{(i,\lambda) \in {}'\Lambda} \cup U$ *is an* R*-free basis of* F_2 .

Let us notice that

$$\{[i,\lambda]\}_{(i,\lambda) \in \tilde{\Lambda}} \cup \{\langle v_{(i,\lambda)} \rangle\}_{(i,\lambda) \in {}'\Lambda} \cup \langle U \rangle$$

is an R-free basis of ${}'F_2$. We define ${}^*F_2$ to be the direct summand of ${}'F_2$ generated by

$$\{[i,\lambda]\}_{(i,\lambda) \in \tilde{\Lambda}} \cup \langle U \rangle \, .$$

Let ${}^*\varphi_2$ be the restriction of ${}'\varphi_2$ to ${}^*F_2$. We need the next result at the final step of the process to compute symbolic powers.

Theorem 3.4 *If we can take* $\tilde{\Lambda}$ *itself as* ${}'\Lambda$ *, then*

$$0 \longrightarrow {}^*F_2 \xrightarrow{{}^*\varphi_2} {}^*F_1 \xrightarrow{{}^*\varphi_1} {}^*F_0 = R$$

is acyclic. Hence we have $\operatorname{depth} R/(\mathfrak{a} :_R Q) > 0$.

Proof. If ${}'\Lambda = \tilde{\Lambda}$, there exists a homomorphism $\phi : {}'F_2 \longrightarrow {}'F_3$ such that

$$\begin{aligned} \phi\,([i,\lambda]) &= 0 \quad \text{for any } (i,\lambda) \in \tilde{\Lambda} \, , \\ \phi\,(\langle v_{(i,\lambda)} \rangle) &= (-1)^i \cdot \check{e}_i \otimes w_\lambda \quad \text{for any } (i,\lambda) \in \tilde{\Lambda} \, , \\ \phi\,(\langle u \rangle) &= 0 \quad \text{for any } u \in U \, . \end{aligned}$$

Then $\phi \circ {}'\varphi_3 = \mathrm{id}_{{}'F_3}$ and $\operatorname{Ker} \phi = {}^*F_2$. Hence by (1) of 2.3 we get the required assertion.

In the rest of this section, we assume ${}'\Lambda \subsetneq \tilde{\Lambda}$ and put ${}^*\Lambda = \tilde{\Lambda} \setminus {}'\Lambda$. Then, for any $(j,\mu) \in {}^*\Lambda$, it is possible to write

$$v_{(j,\mu)} = \sum_{(i,\lambda) \in {}'\Lambda} a^{(j,\mu)}_{(i,\lambda)} \cdot v_{(i,\lambda)} + \sum_{u \in U} b^{(j,\mu)}_u \cdot u \, ,$$

where $a^{(j,\mu)}_{(i,\lambda)} , b^{(j,\mu)}_u \in R$. Here, if ${}'\Lambda$ is big enough, we can choose every $b^{(j,\mu)}_u$ from $\mathfrak{m}$. In fact, if $b^{(j,\mu)}_u \notin \mathfrak{m}$ for some $u \in U$, then we can replace ${}'\Lambda$ and U by ${}'\Lambda \cup \{ (j,\mu) \}$ and $U \setminus \{ u \}$, respectively. Furthermore, because of a practical reason, let us allow that some terms of $v_{(i,\lambda)}$ for $(i,\lambda) \in {}^*\Lambda$ with non-unit coefficients appear in the right hand side, that is, for any $(j,\mu) \in {}^*\Lambda$, we write

$$v_{(j,\mu)} = \sum_{(i,\lambda) \in \tilde{\Lambda}} a^{(j,\mu)}_{(i,\lambda)} \cdot v_{(i,\lambda)} + \sum_{u \in U} b^{(j,\mu)}_u \cdot u \, ,$$

where

$$a^{(j,\mu)}_{(i,\lambda)} \in \begin{cases} R & \text{if } (i,\lambda) \in {}'\Lambda \\ \mathfrak{m} & \text{if } (i,\lambda) \in \tilde{\Lambda} \end{cases} \qquad \text{and} \quad b^{(j,\mu)}_u \in \mathfrak{m} \, .$$

Using this expression, for any $(j,\mu) \in {}^*\Lambda$, the following element in ${}'F_3$ can be defined.

$${}^*w_{(j,\mu)} := (-1)^j \cdot \check{e}_j \otimes w_\mu + \sum_{(i,\lambda) \in \tilde{\Lambda}} (-1)^{i+1} a^{(j,\mu)}_{(i,\lambda)} \cdot \check{e}_i \otimes w_\lambda \, .$$

Lemma 3.5 *For any $i \in \{1,2,3\}$, let s_i and t_i be integers such that $s_i < t_i$ and $\{1,2,3\} = \{i, s_i, t_i\}$. Then, for any $(j,\mu) \in {}^*\Lambda$, we have*

$$
\begin{aligned}
{}'\varphi_3({}^*w_{(j,\mu)}) = (-1)^j x_{s_j} \cdot [t_j, \mu] + (-1)^{j+1} x_{t_j} \cdot [s_j, \mu] \\
+ \sum_{(i,\lambda) \in \tilde{\Lambda}} (-1)^{i+1} a^{(j,\mu)}_{(i,\lambda)} \cdot \{x_{s_i} \cdot [t_i, \lambda] - x_{t_i} \cdot [s_i, \lambda]\} + \sum_{u \in U} b^{(j,\mu)}_u \cdot \langle u \rangle \,.
\end{aligned}
$$

As a consequence, we have ${}'\varphi_3({}^*w_{(j,\mu)}) \in \mathfrak{m}\,{}^*F_2$ for any $(j,\mu) \in {}^*\Lambda$.

Proof. By the definition of ${}'\varphi_3$, for any $(j,\mu) \in {}^*\Lambda$, we have

$$
{}'\varphi_3({}^*w_{(j,\mu)}) = [(\partial_2 \otimes \mathrm{id}_{F_3})({}^*w_{(j,\mu)})] + \langle \sigma_2({}^*w_{(j,\mu)}) \rangle \,.
$$

Because

$$
\begin{aligned}
(\partial_2 \otimes \mathrm{id}_{F_3})({}^*w_{(j,\mu)}) &= (-1)^j \cdot \partial_2 \check{e}_j \otimes w_\mu + \sum_{(i,\lambda) \in \tilde{\Lambda}} (-1)^{i+1} a^{(j,\mu)}_{(i,\lambda)} \cdot \partial_2 \check{e}_i \otimes w_\lambda \\
&= (-1)^j \cdot \partial_2 (e_{s_j} \wedge e_{t_j}) \otimes w_\mu + \sum_{(i,\lambda) \in \tilde{\Lambda}} (-1)^{i+1} a^{(j,\mu)}_{(i,\lambda)} \cdot \partial_2 (e_{s_i} \wedge e_{t_i}) \otimes w_\lambda \\
&= (-1)^j \cdot (x_{s_j} e_{t_j} - x_{t_j} e_{s_j}) \otimes w_\mu \\
&\qquad + \sum_{(i,\lambda) \in \tilde{\Lambda}} (-1)^{i+1} a^{(j,\mu)}_{(i,\lambda)} \cdot (x_{s_i} e_{t_i} - x_{t_i} e_{s_i}) \otimes w_\lambda
\end{aligned}
$$

and

$$
\begin{aligned}
\sigma_2({}^*w_{(j,\mu)}) &= (-1)^j \cdot \sigma_2(\check{e}_j \otimes w_\mu) + \sum_{(i,\lambda) \in \tilde{\Lambda}} (-1)^{i+1} a^{(j,\mu)}_{(i,\lambda)} \cdot \sigma_2(\check{e}_i \otimes w_\lambda) \\
&= v_{(j,\mu)} - \sum_{(i,\lambda) \in \tilde{\Lambda}} a^{(j,\mu)}_{(i,\lambda)} \cdot v_{(i,\lambda)} \\
&= \sum_{u \in U} b^{(j,\mu)}_u \cdot u \,,
\end{aligned}
$$

we get the required equality.

Let ${}^*F_3$ be the R-submodule of ${}'F_3$ generated by $\{ {}^*w_{(j,\mu)} \}_{(j,\mu) \in {}^*\Lambda}$ and let ${}^*\varphi_3$ be the restriction of ${}'\varphi_3$ to ${}^*F_3$. By 3.5 we have $\mathrm{Im}\,{}^*\varphi_3 \subseteq {}^*F_2$. Thus we get a complex

$$
0 \longrightarrow {}^*F_3 \xrightarrow{{}^*\varphi_3} {}^*F_2 \xrightarrow{{}^*\varphi_2} {}^*F_1 \xrightarrow{{}^*\varphi_1} {}^*F_0 = R \,.
$$

This is the complex we desire. In fact, the following result holds.

Theorem 3.6 $({}^*F_\bullet, {}^*\varphi_\bullet)$ *is an acyclic complex of finitely generated free R-modules with the following properties.*

(1) $\mathrm{Im}\,{}^*\varphi_1 = \mathfrak{a} :_R Q$ *and* $\mathrm{Im}\,{}^*\varphi_3 \subseteq \mathfrak{m}\,{}^*F_2$.

(2) $\{\, {}^*w_{(j,\mu)}\,\}_{(j,\mu)\,\in\,{}^*\Lambda}$ *is an R-free basis of* ${}^*F_3$.

(3) $\{\,[i,\lambda]\,\}_{(i,\lambda)\,\in\,\tilde{\Lambda}} \cup \langle U\rangle$ *is an R-free basis of* ${}^*F_2$.

Proof. First, let us notice that $\{\check{e}_i \otimes w_\lambda\}_{(i,\lambda)\,\in\,\tilde{\Lambda}}$ is an R-free basis of ${}'F_3$ and

$$\check{e}_j \otimes w_\mu \in R\cdot{}^*w_{(j,\mu)} + R\cdot\{\check{e}_i \otimes w_\lambda\}_{(i,\lambda)\,\in\,{}'\Lambda} + \mathfrak{m}\,{}'F_3$$

for any $(j,\mu) \in {}^*\Lambda$. Hence, by Nakayama's lemma it follows that ${}'F_3$ is generated by

$$\{\check{e}_i \otimes w_\lambda\}_{(i,\lambda)\,\in\,{}'\Lambda} \cup \{{}^*w_{(j,\mu)}\}_{(j,\mu)\,\in\,{}^*\Lambda}\,,$$

which must be an R-free basis since $\mathrm{rank}_R\,{}'F_3 = |\,\tilde{\Lambda}\,| = |\,{}'\Lambda\,| + |\,{}^*\Lambda\,|$. Let ${}''F_3$ be the R-submodule of ${}'F_3$ generated by $\{\,\check{e}_i \otimes w_\lambda\,\}_{(i,\lambda)\,\in\,{}'\Lambda}$. Then ${}''F_3 \oplus {}^*F_3$.

Next, let us recall that

$$\{[i,\lambda]\}_{(i,\lambda)\,\in\,\tilde{\Lambda}} \cup \{\langle v_{(i,\lambda)}\rangle\}_{(i,\lambda)\,\in\,{}'\Lambda} \cup \langle U\rangle$$

is an R-free basis of ${}'F_2$. Because

$$\langle v_{(i,\lambda)}\rangle = (-1)^i\cdot{}'\varphi_3(\check{e}_i \otimes w_\lambda) + (-1)^i\cdot[\partial_2\check{e}_i \otimes w_\lambda]\,,$$

we see that

$$\{[i,\lambda]\}_{(i,\lambda)\,\in\,\tilde{\Lambda}} \cup \{{}'\varphi_3(\check{e}_i \otimes w_\lambda)\}_{(i,\lambda)\,\in\,{}'\Lambda} \cup \langle U\rangle$$

is also an R-free basis. Let ${}''F_2 = R\cdot\{\,{}'\varphi_3(\check{e}_i \otimes w_\lambda)\,\}_{(i,\lambda)\,\in\,{}'\Lambda}$. Then ${}'F_2 = {}''F_2 \oplus {}^*F_2$.

It is obvious that ${}'\varphi_3({}''F_3) = {}''F_2$. Moreover, by 3.5 we get ${}'\varphi_3({}^*F_3) \subseteq {}^*F_2$. Therefore, by (2) of 2.3, it follows that ${}^*F_\bullet$ is acyclic. We have already seen (3) and the first assertion of (1). The second assertion of (1) follows from 3.5. Moreover, the assertion (2) is now obvious.

4 Computing symbolic powers

Let x, y, z be an sop for R and I an ideal of R generated by the maximal minors of the matrix

$$\Phi = \begin{pmatrix} x^\alpha & y^\beta & z^\gamma \\ y^{\beta'} & z^{\gamma'} & x^{\alpha'} \end{pmatrix},$$

where $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ are positive integers. As is well known, R/I is a Cohen-Macaulay ring with $\dim R/I = 1$. In this section, we give a minimal free resolution of I^n for any $n > 0$, and consider its $*$-transform in order to compute the symbolic power $I^{(n)}$. We put

$$a = z^{\gamma+\gamma'} - x^{\alpha'}y^\beta\,,\; b = x^{\alpha+\alpha'} - y^{\beta'}z^\gamma\,,\; c = y^{\beta+\beta'} - x^\alpha z^{\gamma'}\,.$$

Then, $I = (a,b,c)R$ and we have the next result (See [3] for the definition of d-sequences).

Lemma 4.1 *The following assertions hold.*

(1) $x^\alpha a + y^\beta b + z^\gamma c = 0$ *and* $y^{\beta'}a + z^{\gamma'}b + x^{\alpha'}c = 0$.

(2) *Let* $\mathfrak{p} \in \operatorname{Ass}_R(R/I)$. *Then* $IR_{\mathfrak{p}}$ *is generated by any two elements of* a, b, c.

(3) *Any two elements of* a, b, c *form an ssop for* R.

(4) a, b, c *is an unconditioned d-sequence.*

Proof. (1) These equalities can be checked directly.

(2) Let us prove $IR_{\mathfrak{p}} = (a, b)R_{\mathfrak{p}}$. If $x \in \mathfrak{p}$, then $y, z \in \sqrt{(a, c, x)R} \subseteq \mathfrak{p}$, and so $\mathfrak{p} = \mathfrak{m}$, which contradicts to the Cohen-Macaulayness of R/I. Hence $x \notin \mathfrak{p}$. Then

$$c = -(y^{\beta'}a + z^{\gamma'}b)/x^{\alpha'} \in (a, b)R_{\mathfrak{p}},$$

which means $IR_{\mathfrak{p}} = (a, b)R_{\mathfrak{p}}$.

(3) For example, as $x, z \in \sqrt{(a, b, y)R}$, it follows that a, b is an ssop for R.

(4) Let us prove that a, b, c is a d-sequence. As a, b is a regular sequence by (3), it is enough to show $(a, b)R :_R c^2 = (a, b)R :_R c$. We obviously have $(a, b)R :_R c^2 \supseteq (a, b)R :_R c$. Take any $\mathfrak{q} \in \operatorname{Ass}_R(R/(a, b)R :_R c)$. As $R/(a, b)R :_R c \hookrightarrow R/(a, b)R$, we have $\operatorname{ht}_R \mathfrak{q} = 2$. If $c \in \mathfrak{q}$, then $\mathfrak{q} \in \operatorname{Min}_R(R/I)$, and so $IR_{\mathfrak{q}} = (a, b)R_{\mathfrak{q}}$ by (2), which means

$$(a, b)R_{\mathfrak{q}} :_{R_{\mathfrak{q}}} c^2 = (a, b)R_{\mathfrak{q}} :_{R_{\mathfrak{q}}} c = R_{\mathfrak{q}}.$$

If $c \notin \mathfrak{q}$, we have

$$(a, b)R_{\mathfrak{q}} :_{R_{\mathfrak{q}}} c^2 = (a, b)R_{\mathfrak{q}} :_{R_{\mathfrak{q}}} c = (a, b)R_{\mathfrak{q}}.$$

Therefore we get the required equality.

We take an indeterminate t and consider the Rees algebra $R[It]$. Moreover, we take three indeterminates A, B, C and put $S = R[A, B, C]$. We regard S as a $\mathbb{Z}$-graded ring by setting $\deg A = \deg B = \deg C = 1$. Let $\pi : S \longrightarrow R[It]$ be the graded homomorphism of R-algebras such that $\pi(A) = at$, $\pi(B) = bt$ and $\pi(C) = ct$. By (4) of 4.1 it follows that $\operatorname{Ker} \pi$ is generated by linear forms (cf. [4, Theorem 3.1]). On the other hand,

$$0 \longrightarrow R^{\oplus 2} \xrightarrow{{}^t\Phi} R^{\oplus 3} \xrightarrow{(a\ \ b\ \ c)} R \longrightarrow R/I \longrightarrow 0$$

is a minimal free resolution of R/I. Hence $\operatorname{Ker} \pi$ is generated by

$$f := x^{\alpha}A + y^{\beta}B + z^{\gamma}C \quad \text{and} \quad g := y^{\beta'}A + z^{\gamma'}B + x^{\alpha'}C.$$

Thus we get $S/(f, g)S \cong R[It]$. Then, as f, g is a regular sequence of S,

$$0 \longrightarrow S(-2) \xrightarrow{\binom{-g}{f}} S(-1) \oplus S(-1) \xrightarrow{(f\ \ g)} S \xrightarrow{\pi} R[It] \longrightarrow 0$$

is a graded S-free resolution of $R[It]$. Now we take its homogeneous part of degree n, and get the next result.

Theorem 4.2 *For any* $n \geq 2$,

$$0 \longrightarrow S_{n-2} \xrightarrow{\binom{-g}{f}} S_{n-1} \oplus S_{n-1} \xrightarrow{(f\ \ g)} S_n \xrightarrow{\epsilon} R$$

is acyclic and it is a minimal free resolution of I^n, *where* S_d $(d \in \mathbb{Z})$ *is the* R*-submodule of* S *consisting of homogeneous elements of degree* d *and* ϵ *is the* R*-linear map defined by substituting* a, b, c *for* A, B, C, *respectively.*

Let us denote the complex in 4.2 by $(F^1_\bullet, \varphi^1_\bullet)$, that is, we set

$$F^1_3 = S_{n-2}\,,\ F^1_2 = S_{n-1} \oplus S_{n-1}\,,\ F^1_1 = S_n\,,\ F^1_0 = R\,,$$
$$\varphi^1_3 = \left(\begin{smallmatrix} -g \\ f \end{smallmatrix}\right),\ \varphi^1_2 = (f\ \ g) \quad \text{and} \quad \varphi^1_1 = \epsilon\,.$$

Then $F^1_\bullet$ is an acyclic complex of finitely generated free R-modules and $\operatorname{Im}\varphi^1_1 = I^n$. The number "1" of $F^1_\bullet$ means that it is the first acyclic complex we need for computing $I^{(n)}$. Our strategy is as follows. Taking the $*$-transform of $F^1_\bullet$ with respect to suitable powers of x, y, z, we get ${}^*F^1_\bullet$, which is denoted by $F^2_\bullet$. If its length is still 3, we again take some $*$-transform of $F^2_\bullet$ and get $F^3_\bullet$. By repeating this operation successively, we eventually get an acyclic complex $F^k_\bullet$ of length 2. Then the family $\{\, F^i_\bullet \,\}_{1\le i\le k}$ of acyclic complexes has the complete information on $I^{(n)}$.

Let $\alpha'' = \min\{\alpha, \alpha'\}$, $\beta'' = \min\{\beta, \beta'\}$, $\gamma'' = \min\{\gamma, \gamma'\}$ and $Q = (x^{\alpha''}, y^{\beta''}, z^{\gamma''})R$. Because f and g are elements of Q, we have $\operatorname{Im}\varphi^1_3 \subseteq QF^1_2$, and so by 3.1 we get the following.

Theorem 4.3 $(I^n :_R Q)/I^n \cong (R/Q)^{\oplus\binom{n}{2}}$.

Now we are going to take the $*$-transform of $F^1_\bullet$ with respect to $x^{\alpha''}$, $y^{\beta''}$, $z^{\gamma''}$. At first, we have to fix Λ^1, which is an R-free basis of F^1_3. For any $0 \le d \in \mathbb{Z}$, let us denote by $\mathrm{m}^d_{A,B,C}$ the set $\{A^iB^jC^k \mid 0 \le i,j,k \in \mathbb{Z} \text{ and } i+j+k = d\}$, which is an R-free basis of S_d. We take $\mathrm{m}^{n-2}_{A,B,C}$ as Λ^1. Then, for any $M \in \mathrm{m}^{n-2}_{A,B,C}$, we have to write

$$\varphi^1_3(M) = x^{\alpha''}\cdot v^1_{(1,M)} + y^{\beta''}\cdot v^1_{(2,M)} + z^{\gamma''}\cdot v^1_{(3,M)}\,,$$

where $v^1_{(i,M)} \in F^1_2$ for $i = 1,2,3$. As is described at the end of Introduction, for $h \in S_{n-1}$, let us denote the elements

$$\begin{pmatrix} h \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ h \end{pmatrix} \in F^1_2 = S_{n-1} \oplus S_{n-1}$$

by $[h]$ and $\langle h\rangle$, respectively. Then, for any $M \in \mathrm{m}^{n-2}_{A,B,C}$, we have

$$\begin{aligned}
\varphi^1_3(M) &= \begin{pmatrix} -gM \\ fM \end{pmatrix} \\
&= \begin{pmatrix} -y^{\beta'}AM - z^{\gamma'}BM - x^{\alpha'}CM \\ x^{\alpha}AM + y^{\beta}BM + z^{\gamma}CM \end{pmatrix} \\
&= -y^{\beta'}\cdot[AM] - z^{\gamma'}\cdot[BM] - x^{\alpha'}\cdot[CM] \\
&\qquad\quad + x^{\alpha}\cdot\langle AM\rangle + y^{\beta}\cdot\langle BM\rangle + z^{\gamma}\cdot\langle CM\rangle \\
&= x^{\alpha''}\cdot v^1_{(1,M)} + y^{\beta''}\cdot v^1_{(2,M)} + z^{\gamma''}\cdot v^1_{(3,M)}\,,
\end{aligned}$$

where

$$\begin{aligned}
v^1_{(1,M)} &= x^{\alpha-\alpha''}\cdot\langle AM\rangle - x^{\alpha'-\alpha''}\cdot[CM]\,, \\
v^1_{(2,M)} &= y^{\beta-\beta''}\cdot\langle BM\rangle - y^{\beta'-\beta''}\cdot[AM]\,, \\
v^1_{(3,M)} &= z^{\gamma-\gamma''}\cdot\langle CM\rangle - z^{\gamma'-\gamma''}\cdot[BM]\,.
\end{aligned}$$

We set $\widetilde{\Lambda}^1 = \{1,2,3\} \times \Lambda^1$ and we have to choose its subset $'\Lambda^1$ as big as possible so that $\{v^1_{(i,M)}\}_{(i,M)\in '\Lambda^1}$ is a part of an R-free basis of F^1_2 . For that purpose, we need to fix a canonical R-free basis of F^1_2 . For a subset H of S_{n-1} , we denote the families $\{[h]\}_{h\in H}$ and $\{\langle h\rangle\}_{h\in H}$ by $[H]$ and $\langle H\rangle$, respectively. Let us notice that $[\mathrm{m}^{n-1}_{A,B,C}] \cup \langle \mathrm{m}^{n-1}_{A,B,C}\rangle$ is an R-free basis of F^1_2 .

Setting $n = 2$, we get the next result.

Theorem 4.4 (cf. [2]) *$I^{(2)} = I^2 :_R Q$ and $I^{(2)}/I^2 \cong R/Q$.*

Proof. By replacing rows and columns of Φ and by replacing x, y, z if necessary, we may assume that one of the following conditions are satisfied;

$$\text{(i)}\, \alpha \le \alpha' \,,\ \beta \le \beta' \,,\ \gamma \le \gamma' \,; \quad \text{(ii)}\, \alpha \ge \alpha' \,,\ \beta \le \beta' \,,\ \gamma \le \gamma' \,.$$

Let $n = 2$. Then $\Lambda^1 = \{\,1\,\}$. In the case (i), we have $\alpha'' = \alpha, \beta'' = \beta, \gamma'' = \gamma$ and

$$v^1_{(1,1)} = \langle A\rangle - x^{\alpha'-\alpha}\cdot[C] \,,\ v^1_{(2,1)} = \langle B\rangle - y^{\beta'-\beta}\cdot[A] \,,\ v^1_{(3,1)} = \langle C\rangle - z^{\gamma'-\gamma}\cdot[B] \,.$$

Then, as $[\mathrm{m}^1_{A,B,C}] \cup \langle \mathrm{m}^1_{A,B,C}\rangle$ is an R-free basis, by 3.3 we see that $\{v^1_{(i,1)}\}_{i=1,2,3} \cup [\mathrm{m}^1_{A,B,C}]$ is an R-free basis of F^1_2 . On the other hand, in the case (ii), we have $\alpha'' = \alpha', \beta'' = \beta, \gamma'' = \gamma$ and

$$v^1_{(1,1)} = x^{\alpha-\alpha'}\cdot\langle A\rangle - [C] \,,\ v^1_{(2,1)} = \langle B\rangle - y^{\beta'-\beta}\cdot[A] \,,\ v^1_{(3,1)} = \langle C\rangle - z^{\gamma'-\gamma}\cdot[B] \,.$$

Then, $\{v^1_{(i,1)}\}_{i=1,2,3} \cup \{\langle A\rangle, [A], [B]\}$ is an R-free basis of F^1_2 . In either case, we can take $\widetilde{\Lambda}^1$ as $'\Lambda^1$. Hence, by 3.4 we see depth $R/(I^2 :_R Q) > 0$, and so $I^{(2)} = I^2 :_R Q$. The second assertion follows from 4.3.

Similarly as the proof of 4.4, in order to study $I^{(n)}$ for $n \ge 3$, we have to consider dividing the situation into several cases. In the rest of this section, let us assume

$$\alpha = 1 \,,\ \alpha' = 2 \,,\ 2\beta \le \beta' \,,\ 2\gamma \le \gamma' \,,$$

and explain how to compute $I^{(3)}$ using $*$-transforms. We have $\alpha'' = 1, \beta'' = \beta$ and $\gamma'' = \gamma$. Let $n = 3$. Then $\Lambda^1 = \{A, B, C\}$ and we have

$$\begin{aligned}
v^1_{(1,A)} &= \langle A^2\rangle - x\cdot[AC] \,, & v^1_{(2,A)} &= \langle AB\rangle - y^{\beta'-\beta}\cdot[A^2] \,, & v^1_{(3,A)} &= \langle AC\rangle - z^{\gamma'-\gamma}\cdot[AB] \,,\\
v^1_{(1,B)} &= \langle AB\rangle - x\cdot[BC] \,, & v^1_{(2,B)} &= \langle B^2\rangle - y^{\beta'-\beta}\cdot[AB] \,, & v^1_{(3,B)} &= \langle BC\rangle - z^{\gamma'-\gamma}\cdot[B^2] \,,\\
v^1_{(1,C)} &= \langle AC\rangle - x\cdot[C^2] \,, & v^1_{(2,C)} &= \langle BC\rangle - y^{\beta'-\beta}\cdot[AC] \,, & v^1_{(3,C)} &= \langle C^2\rangle - z^{\gamma'-\gamma}\cdot[BC] \,.
\end{aligned}$$

We set $'\Lambda^1 = \{(1,A)\,, (2,A)\,, (3,A)\,, (2,B)\,, (3,B)\,, (3,C)\} \subseteq \widetilde{\Lambda}^1 = \{1,2,3\} \times \{A,B,C\}$. Then we have the following.

Lemma 4.5 *$\{v^1_{(i,M)}\}_{(i,M)\in '\Lambda^1} \cup [\mathrm{m}^2_{A,B,C}]$ is an R-free basis of F^1_2 .*

Proof. Let us recall that $[\mathrm{m}^2_{A,B,C}] \cup \langle \mathrm{m}^2_{A,B,C} \rangle$ is an R-free basis of F^1_2. Because $|{'\Lambda^1}| = |\mathrm{m}^2_{A,B,C}| = 6$ and $\langle \mathrm{m}^2_{A,B,C} \rangle \subseteq R{\cdot}\{v^1_{(i,M)}\}_{(i,M) \in {'\Lambda^1}} + R{\cdot}[\mathrm{m}^2_{A,B,C}]$, we get the required assertion by 3.3.

Let $K_\bullet = K_\bullet(x, y^\beta, z^\gamma)$. By 3.2 there exists a chain map $\sigma^1_\bullet : K_\bullet \otimes_R F^1_3 \longrightarrow F^1_\bullet$ such that $\operatorname{Im} \sigma^1_0 + \operatorname{Im} \varphi^1_1 = I^3 :_R Q$ and $\sigma^1_2(\check{e}_i \otimes M) = (-1)^i \cdot v^1_{(i,M)}$ for any $(i, M) \in \widetilde{\Lambda^1} = \{1,2,3\} \times \{A, B, C\}$. Moreover, we get an acyclic complex

$$0 \longrightarrow {'F^1_3} \xrightarrow{{'\varphi^1_3}} {'F^1_2} \xrightarrow{{'\varphi^1_2}} {^*F^1_1} \xrightarrow{{^*\varphi^1_1}} {^*F^1_0} = R\,,$$

where

$${'F^1_3} = K_2 \otimes_R F^1_3\,, \quad {'F^1_2} = \begin{matrix} K_1 \otimes_R F^1_3 \\ \oplus \\ F^1_2 \end{matrix}\,, \quad {^*F^1_1} = \begin{matrix} K_0 \otimes_R F^1_3 \\ \oplus \\ F^1_1 \end{matrix}\,, \quad {'\varphi^1_3} = \begin{pmatrix} \partial_2 \otimes \mathrm{id}_{F^1_3} \\ \sigma^1_2 \end{pmatrix},$$

and ${^*\varphi^1_1} = (\,\sigma^1_0 \;\; \varphi^1_1\,)$. Let us recall our notation introduced in Section 3. For any $(i, M) \in \widetilde{\Lambda^1}$ we set

$$[i, M] = [e_i \otimes M] = \begin{pmatrix} e_i \otimes M \\ 0 \end{pmatrix} \in {'F^1_3}\,.$$

On the other hand, for any $\eta \in F^1_2$, we set

$$\langle \eta \rangle = \begin{pmatrix} 0 \\ \eta \end{pmatrix} \in {'F^1_2}\,.$$

In particular, as $F^1_2 = S_2 \oplus S_2$, $\langle [h] \rangle \in {'F^1_2}$ is defined for any $h \in S_2$. We set $\langle [\mathrm{m}^2_{A,B,C}] \rangle = \{\langle [M] \rangle\}_{M \in \mathrm{m}^2_{A,B,C}}$. Let ${^*F^1_2}$ be the R-submodule of ${'F^1_2}$ generated by

$$\{[i, M]\}_{(i,M) \in \widetilde{\Lambda^1}} \cup \langle [\mathrm{m}^2_{A,B,C}] \rangle\,,$$

and let ${^*\varphi^1_2}$ be the restriction of ${'\varphi^1_2}$ to ${^*F^1_2}$. In order to define ${^*F^1_3}$, we set ${^*\Lambda^1} = \widetilde{\Lambda^1} \setminus {'\Lambda^1} = \{(1, B)\,, (1, C)\,, (2, C)\}$. We need the next result which can be checked directly.

Lemma 4.6 *The following equalities hold;*

$$\begin{aligned} v^1_{(1,B)} &= v^1_{(2,A)} - x{\cdot}[BC] + y^{\beta'-\beta}{\cdot}[A^2]\,, \\ v^1_{(1,C)} &= v^1_{(3,A)} - x{\cdot}[C^2] + z^{\gamma'-\gamma}{\cdot}[AB]\,, \\ v^1_{(2,C)} &= v^1_{(3,B)} - y^{\beta'-\beta}{\cdot}[AC] + z^{\gamma'-\gamma}{\cdot}[B^2]\,. \end{aligned}$$

So, we define the elements ${^*w^1_{(i,M)}} \in {'F^1_3}$ for $(i, M) \in {^*\Lambda^1}$ as follows;

$$\begin{aligned} {^*w^1_{(1,B)}} &= -\check{e}_1 \otimes B - \check{e}_2 \otimes A\,, \\ {^*w^1_{(1,C)}} &= -\check{e}_1 \otimes C + \check{e}_3 \otimes A\,, \\ {^*w^1_{(2,C)}} &= \check{e}_2 \otimes C + \check{e}_3 \otimes B\,. \end{aligned}$$

Let ${}^*F_3^1$ be the R-submodule of ${}'F_3^1$ generated by $\{{}^*w_{(i,M)}^1\}_{(i,M)\in {}^*\Lambda^1}$ and let ${}^*\varphi_3^1$ be the restriction of ${}'\varphi_3^1$ to ${}^*F_3^1$. Thus we get a complex

$$0 \longrightarrow {}^*F_3^1 \xrightarrow{{}^*\varphi_3^1} {}^*F_2^1 \xrightarrow{{}^*\varphi_2^1} {}^*F_1^1 \xrightarrow{{}^*\varphi_1^1} {}^*F_0^1 = R\,.$$

Let us denote $({}^*F_\bullet^1\,, {}^*\varphi_\bullet^1)$ by $(F_\bullet^2\,, \varphi_\bullet^2)$. Moreover, we put $w_{(i,M)}^2 = {}^*w_{(i,M)}^1$ for $(i,M) \in {}^*\Lambda^1$. Then, by 3.5 and 3.6 we have the next result.

Lemma 4.7 *$(F_\bullet^2\,, \varphi_\bullet^2)$ is an acyclic complex of finitely generated free R-modules satisfying the following conditions.*

(1) $\operatorname{Im}\varphi_1^2 = I^3 :_R Q$.

(2) $\{w_{(1,B)}^2\,, w_{(1,C)}^2\,, w_{(2,C)}^2\}$ *is an R-free basis of F_3^2.*

(3) $\{[i,M]\}_{(i,M)\in\widetilde{\Lambda^1}} \cup \langle[\mathrm{m}_{A,B,C}^2]\rangle$ *is an R-free basis of F_2^2.*

(4) *The following equalities hold;*

$$\begin{aligned}
\varphi_3^2(w_{(1,B)}^2) &= -y^\beta\cdot[3,B] + z^\gamma\cdot[2,B] - x\cdot[3,A] + z^\gamma\cdot[1,A]\\
&\qquad\qquad\qquad - x\cdot\langle[BC]\rangle + y^{\beta'-\beta}\cdot\langle[A^2]\rangle\,,\\
\varphi_3^2(w_{(1,C)}^2) &= -y^\beta\cdot[3,C] + z^\gamma\cdot[2,C] + x\cdot[2,A] - y^\beta\cdot[1,A]\\
&\qquad\qquad\qquad - x\cdot\langle[C^2]\rangle + z^{\gamma'-\gamma}\cdot\langle[AB]\rangle\,,\\
\varphi_3^2(w_{(2,C)}^2) &= x\cdot[3,C] - z^\gamma\cdot[1,C] + x\cdot[2,B] - y^\beta\cdot[1,B]\\
&\qquad\qquad\qquad - y^{\beta'-\beta}\cdot\langle[AC]\rangle + z^{\gamma'-\gamma}\cdot\langle[B^2]\rangle\,.
\end{aligned}$$

We put $\Lambda^2 = {}^*\Lambda^1 = \{(1,B)\,, (1,C)\,, (2,C)\}$ and $\widetilde{\Lambda^2} = \{1,2,3\}\times\Lambda^2$. We simply denote $(j,(i,M)) \in \widetilde{\Lambda^2}$ by (j,i,M). Then

$$\widetilde{\Lambda^2} = \left\{\begin{array}{lll} (1,1,B)\,, & (2,1,B)\,, & (3,1,B)\,,\\ (1,1,C)\,, & (2,1,C)\,, & (3,1,C)\,,\\ (1,2,C)\,, & (2,2,C)\,, & (3,2,C) \end{array}\right\}.$$

As we are assuming $2\beta \le \beta'$ and $2\gamma \le \gamma'$, by (4) of 4.7 we get

$$\varphi_3^2(w_{(i,M)}^2) = x\cdot\varphi_{(1,i,M)}^2 + y^\beta\cdot\varphi_{(2,i,M)}^2 + z^\gamma\cdot\varphi_{(3,i,M)}^2$$

for any $(i,M) \in \Lambda^2$, where

$v_{(1,1,B)}^2 = -[3,A] - \langle[BC]\rangle\,,\ \ v_{(2,1,B)}^2 = -[3,B] + y^{\beta'-2\beta}\cdot\langle[A^2]\rangle\,,\ \ v_{(3,1,B)}^2 = [2,B] + [1,A]\,,$
$v_{(1,1,C)}^2 = [2,A] - \langle[C^2]\rangle\,,\ \ v_{(2,1,C)}^2 = -[3,C] - [1,A]\,,\ \ v_{(3,1,C)}^2 = [2,C] + z^{\gamma'-2\gamma}\cdot\langle[AB]\rangle\,,$
$v_{(1,2,C)}^2 = [3,C] + [2,B]\,,\ \ v_{(2,2,C)}^2 = -[1,B] - y^{\beta'-2\beta}\cdot\langle[AC]\rangle$ and
$v_{(3,2,C)}^2 = -[1,C] + z^{\gamma'-2\gamma}\cdot\langle[B^2]\rangle\,.$

Thus a family $\{v_{(j,i,M)}^2\}_{(j,i,M)\in\widetilde{\Lambda^2}}$ of elements in F_2^2 is fixed and we see $\operatorname{Im}\varphi_3^2 \subseteq QF_2^2$. Because $\operatorname{Im}\varphi_1^2 :_R Q = (I^3 :_R Q) :_R Q = I^3 :_R Q^2$ and $F_3^2 \cong R^{\oplus 3}$, by 3.1 we get the next result.

Theorem 4.8 *Let $\alpha = 1$, $\alpha' = 2$, $2\beta \leq \beta'$ and $2\gamma \leq \gamma'$. Then we have*

$$(I^3 :_R Q^2)/(I^3 :_R Q) \cong (R/Q)^{\oplus 3}\,.$$

The following relation, which can be checked directly, is very important.

Lemma 4.9 $v^2_{(3,1,B)} + v^2_{(2,1,C)} + v^2_{(1,2,C)} = 2\cdot[2, B]$.

By 3.2 there exists a chain map $\sigma^2_\bullet : K_\bullet \otimes_R F^2_3 \longrightarrow F^2_\bullet$ such that $\operatorname{Im}\sigma^2_0 + \operatorname{Im}\varphi^2_1 = I^3 :_R Q^2$ and $\sigma^2_2(\check{e}_j \otimes w^2_{(i,M)}) = (-1)^j \cdot v^2_{(j,i,M)}$ for any $(j, i, M) \in \widetilde{\Lambda}^2$. Moreover, we get an acyclic complex

$$0 \longrightarrow {}'F^2_3 \xrightarrow{{}'\varphi^2_3} {}'F^2_2 \xrightarrow{{}'\varphi^2_2} {}^*F^2_1 \xrightarrow{{}^*\varphi^2_1} {}^*F^2_0 = R\,,$$

where

$${}'F^2_3 = K_2 \otimes_R F^2_3\,, \quad {}'F^2_2 = \begin{array}{c} K_1 \otimes_R F^2_3 \\ \oplus \\ F^2_2 \end{array}, \quad {}^*F^2_1 = \begin{array}{c} K_0 \otimes_R F^2_3 \\ \oplus \\ F^2_1 \end{array}, \quad {}'\varphi^2_3 = \begin{pmatrix} \partial_2 \otimes \mathrm{id}_{F^2_3} \\ \sigma^2_2 \end{pmatrix},$$

and ${}^*\varphi^2_1 = (\sigma^2_0 \;\; \varphi^2_1)$. In order to remove non-minimal components from ${}'F^2_3$ and ${}'F^2_2$, we would like to choose a subset ${}'\Lambda^2$ of $\widetilde{\Lambda}^2$ as big as possible so that $\{v^2_{(j,i,M)}\}_{(j,i,M)\in {}'\Lambda^2}$ is a part of an R-free basis of F^2_2.

Theorem 4.10 *Let $\alpha = 1$, $\alpha' = 2$, $2\beta \leq \beta'$ and $2\gamma \leq \gamma'$. Suppose that 2 is a unit in R. Then, we can take $\widetilde{\Lambda}^2$ itself as ${}'\Lambda^2$. Hence* depth $R/(I^3 :_R Q^2) > 0$, *and so $I^{(3)} = I^3 :_R Q^2$. Moreover, we have $\ell_R(\, I^{(3)}/I^3\,) = 6\cdot\ell_R(\, R/Q\,)$.*

Proof. We would like to show that

$$\{v^2_{(j,i,M)}\}_{(j,i,M)\,\in\,\widetilde{\Lambda}^2} \cup \langle[\mathrm{m}^2_{A,B,C}]\rangle$$

is an R-free basis of F^2_2. Let us recall that

$$\{[i, M]\}_{(i,M)\,\in\,\widetilde{\Lambda}^1} \cup \langle[\mathrm{m}^2_{A,B,C}]\rangle$$

is an R-free basis of F^2_2. Because $|\widetilde{\Lambda}^1| = |\widetilde{\Lambda}^2|$, we have to prove

$$\{[i, M]\}_{(i,M)\,\in\,\widetilde{\Lambda}^1} \subseteq G := R\cdot\{v^2_{(j,i,M)}\}_{(j,i,M)\,\in\,\widetilde{\Lambda}^2} + R\cdot\langle[\mathrm{m}^2_{A,B,C}]\rangle\,.$$

In fact, we have $[2, A] = v^2_{(1,1,C)} + \langle[C^2]\rangle \in G$. Similarly, we can easily see that $[3, A]$, $[1, B]$, $[3, B]$, $[1, C]$ and $[2, C]$ are included in G. Moreover, as 2 is a unit in R, we have $[2, B] \in G$ by 4.9. Then $[1, A] = v^2_{(3,1,B)} - [2, B] \in G$ and $[3, C] = v^2_{(1,2,C)} - [2, B] \in G$. The last assertion holds since

$$\begin{aligned} \ell_R(\, I^{(3)}/I^3\,) &= \ell_R(\,(I^3 :_R Q^2)/I^3\,) \\ &= \ell_R(\,(I^3 :_R Q^2)/(I^3 :_R Q)\,) + \ell_R(\,(I^3 :_R Q)/I^3\,) \\ &= 3\cdot\ell_R(\, R/Q\,) + 3\cdot\ell_R(\, R/Q\,) \end{aligned}$$

by 4.3 and 4.8. Thus the proof is completed.

In the rest of this section, let us consider the case where $\mathrm{ch}\, R = 2$. In this case, we have

$$v^2_{(3,1,B)} + v^2_{(2,1,C)} + v^2_{(1,2,C)} = 0\,.$$

We set ${}'\Lambda^2 = \widetilde{\Lambda^2} \setminus \{\,(3,1,B)\,\}$. Then, it is easy to see that

$$\{v^2_{(j,i,M)}\}_{(j,i,M)\in{}'\Lambda^2} \cup \{\,[2,B]\,\} \cup \langle[\mathrm{m}^2_{A,B,C}]\rangle$$

is an R-free basis of F^2_2. For any $(j,i,M) \in \widetilde{\Lambda^2}$, let us simply denote $[j,(i,M)] = [e_j \otimes w^2_{(i,M)}] \in {}'F^2_2$ by $[j,i,M]$. Then

$$\{\,[j,i,M]\,\}_{(j,i,M)\in\widetilde{\Lambda^2}} \cup \{\,\langle v^2_{(j,i,M)}\rangle\,\}_{(j,i,M)\in{}'\Lambda^2} \cup \{\,\langle[2,B]\rangle\,\} \cup \langle\langle[\mathrm{m}^2_{A,B,C}]\rangle\rangle$$

is an R-free basis of ${}'F^2_2$. Let ${}^*F^2_2$ be the R-submodule of ${}'F^2_2$ generated by

$$\{\,[j,i,M]\,\}_{(j,i,M)\in\widetilde{\Lambda^2}} \cup \{\,\langle[2,B]\rangle\,\} \cup \langle\langle[\mathrm{m}^2_{A,B,C}]\rangle\rangle$$

and let ${}^*\varphi^2_2$ be the restriction of ${}'\varphi^2_2$ to ${}^*F^2_2$. In order to define ${}^*F^2_3$, we set ${}^*\Lambda^2 = \widetilde{\Lambda^2} \setminus {}'\Lambda^2 = \{(3,1,B)\}$. Because

$$v^2_{(3,1,B)} = -v^2_{(2,1,C)} - v^2_{(1,2,C)}\,,$$

we define ${}^*w^2_{(3,1,B)} \in {}'F^2_3$ to be

$$-\check{e}_3 \otimes w^2_{(1,B)} + \check{e}_2 \otimes w^2_{(1,C)} - \check{e}_1 \otimes w^2_{(2,C)}\,.$$

Let ${}^*F^2_3$ be the R-submodule of ${}'F^2_3$ generated by ${}^*w^2_{(3,1,B)}$ and let ${}^*\varphi^2_3$ be the restriction of ${}'\varphi^2_3$ to ${}^*F^2_3$. Thus we get a complex

$$0 \longrightarrow {}^*F^2_3 \xrightarrow{{}^*\varphi^2_3} {}^*F^2_2 \xrightarrow{{}^*\varphi^2_2} {}^*F^2_1 \xrightarrow{{}^*\varphi^2_1} {}^*F^2_0 = R\,.$$

Let us denote $({}^*F^2_\bullet\,, {}^*\varphi^2_\bullet)$ by $(F^3_\bullet\,, \varphi^3_\bullet)$. Moreover, we put $w^3_{(3,1,B)} = {}^*w^2_{(3,1,B)}$. Then, by 3.5 and 3.6 we have the next result.

Lemma 4.11 *$(F^3_\bullet\,, \varphi^3_\bullet)$ is an acyclic complex of finitely generated free R-modules satisfying the following conditions.*

(1) $\mathrm{Im}\,\varphi^3_1 = I^3 :_R Q^2$.

(2) $w^3_{(3,1,B)}$ *is an R-free basis of F^3_3.*

(3) $\{\,[j,i,M]\,\}_{(j,i,M)\in\widetilde{\Lambda^2}} \cup \{\,\langle[2,B]\rangle\,\} \cup \langle\langle[\mathrm{m}^2_{A,B,C}]\rangle\rangle$ *is an R-free basis of F^3_2.*

(4) *The following equality holds* ;

$$\begin{aligned}\varphi^3_3(w^3_{(3,1,B)}) = -x\cdot[2,1,B] + y^\beta\cdot[1,1,B] + x\cdot[3,1,C]\\ - z^\gamma\cdot[1,1,C] - y^\beta\cdot[3,2,C] + z^\gamma\cdot[2,2,C]\,.\end{aligned}$$

We put $\Lambda^3 = {}^*\Lambda^2 = \{(3,1,B)\}$ and $\widetilde{\Lambda^3} = \{1,2,3\} \times \Lambda^3$. We simply denote $(i,(3,1,B)) \in \widetilde{\Lambda^3}$ by $(i,3,1,B)$. Then $\widetilde{\Lambda^3} = \{\,(i,3,1,B)\,\}_{i=1,2,3}$. By (4) of 4.11 we have

$$\varphi_3^3(w^3_{(3,1,B)}) = x\cdot v^3_{(1,3,1,B)} + y^\beta \cdot v^3_{(2,3,1,B)} + z^\gamma \cdot v^3_{(3,3,1,B)}\,,$$

where

$$v^3_{(1,3,1,B)} = [3,1,C] - [2,1,B]\,, \quad v^3_{(2,3,1,B)} = [1,1,B] - [3,2,C] \quad \text{and} \quad v^3_{(3,3,1,B)} = [2,2,C] - [1,1,C]\,.$$

Thus a family $\{v^3_{(i,3,1,B)}\}_{i=1,2,3}$ of elements in F_2^3 is fixed and we see $\operatorname{Im}\varphi_3^3 \subseteq QF_2^3$. Because

$$\operatorname{Im}\varphi_1^3 :_R Q = (I^3 :_R Q^2) :_R Q = I^3 :_R Q^3$$

and $F_3^3 \cong R$, by 3.1 we get the next result.

Theorem 4.12 *Let $\alpha = 1$, $\alpha' = 2$, $2\beta \le \beta'$, $2\gamma \le \gamma'$ and $\operatorname{ch} R = 2$. Then we have*

$$(I^3 :_R Q^3)/(I^3 :_R Q^2) \cong R/Q\,.$$

It is easy to see that

$$\{v^2_{(i,3,1,B)}\}_{i=1,2,3} \cup \left\{ \begin{array}{ll} [2,1,B]\,, & [3,1,B]\,, \\ [1,1,C]\,, & [2,1,C]\,, \\ [1,2,C]\,, & [3,2,C] \end{array} \right\} \cup \langle\langle [\mathrm{m}^2_{A,B,C}] \rangle\rangle$$

is an R-free basis of F_2^3. Therefore, by 3.4 we get the following result.

Theorem 4.13 *Let $\alpha = 1$, $\alpha' = 2$, $2\beta \le \beta'$, $2\gamma \le \gamma'$ and $\operatorname{ch} R = 2$. Then, it follows that* depth $R/(I^3 :_R Q^3) > 0$, *and so $I^{(3)} = I^3 :_R Q^3$. Moreover, we have $\ell_R(\,I^{(3)}/I^3\,) = 7\cdot\ell_R(\,R/Q\,) = 7\alpha\beta\cdot\ell_R(\,R/(x,y,z)R\,)$.*

The last assertion of 4.13 holds since $\ell_R(\,I^{(3)}/I^3\,)$ coincides with

$$\begin{aligned} & \ell_R(\,(I^3 :_R Q^3)/(I^3 :_R Q^2)\,) + \ell_R(\,(I^3 :_R Q^2)/(I^3 :_R Q)\,) + \ell_R(\,(I^3 :_R Q)/I^3\,) \\ = \;& \ell_R(\,R/Q\,) + 3\cdot\ell_R(\,R/Q\,) + 3\cdot\ell_R(\,R/Q\,) \end{aligned}$$

by 4.3, 4.8 and 4.12.

5 Computing ϵ-multiplicity

Let R be a 3-dimensional regular local ring with the maximal ideal $\mathfrak{m} = (x,y,z)R$. Let I be the ideal generated by the maximal minors of the matrix

$$\begin{pmatrix} x & y & z \\ y & z & x^2 \end{pmatrix}.$$

We will compute the length of $I^{(n)}/I^n$ for all n using the $*$-transforms. As a consequence of our result, we get $\epsilon(I) = 1/2$, where

$$\epsilon(I) := \lim_{n\to\infty} \frac{3!}{n^3}\cdot\ell_R(\, I^{(n)}/I^n\,)\,,$$

which is the invariant called ϵ-multiplicity of I (cf. [1], [6]). Let us maintain the same notations as in Section 4. So, $a = z^2 - x^2y$, $b = x^3 - yz$, $c = y^2 - xz$, $S = R[A, B, C]$, $f = xA+yB+zC$ and $g = yA+zB+x^2C$. Furthermore, we need the following notations which is not used in Section 4. For any $0 \le d \in \mathbb{Z}$, we denote the set $\{B^\beta C^\gamma \mid 0 \le \beta\,, \gamma \in \mathbb{Z}$ and $\beta + \gamma = d\}$ by $\mathrm{m}^d_{B,C}$, and for a monomial L and a set $\mathcal{S}$ consisting of monomials, we denote the set $\{LM \mid M \in \mathcal{S}\}$ by $L\cdot\mathcal{S}$.

Let $n \ge 2$. Then the complex

$$0 \longrightarrow S_{n-2} \xrightarrow{\binom{-g}{f}} S_{n-1} \oplus S_{n-1} \xrightarrow{(f\ g)} S_n \xrightarrow{\epsilon} R$$

is acyclic and gives a minimal free resolution of I^n. We denote this complex by $(F^1_\bullet\,, \varphi^1_\bullet)$.

As Λ^1, which is an R-free basis of $F^1_3 = S_{n-2}$, we take $\mathrm{m}^{n-2}_{A,B,C}$. Then, for any $M \in \Lambda^1$, we have

$$\varphi^1_3(M) = x\cdot v^1_{(1,M)} + y\cdot v^1_{(2,M)} + z\cdot v^1_{(3,M)}\,,$$

where

$$v^1_{(1,M)} = \langle AM\rangle - x\cdot[CM]\,,\ \ v^1_{(2,M)} = \langle BM\rangle - [AM]\,,\ \ v^1_{(3,M)} = \langle CM\rangle - [BM]\,.$$

Hence, by 3.1 we get the following.

Lemma 5.1 $(I^n :_R \mathfrak{m})/I^n \cong (R/\mathfrak{m})^{\oplus\binom{n}{2}}$.

The next result can be checked directly.

Lemma 5.2 *Suppose $n \ge 4$. Then, for any $N \in \mathrm{m}^{n-4}_{A,B,C}$, we have*

$$v^1_{(3,A^2N)} = v^1_{(2,ABN)} - v^1_{(1,B^2N)} + v^1_{(1,ACN)} - x\cdot v^1_{(2,C^2N)} + x\cdot v^1_{(3,BCN)}\,.$$

Let $\widetilde{\Lambda^1} = \{1, 2, 3\} \times \Lambda^1$. Then the following result holds.

Lemma 5.3 *As an R-module, $F^1_2 = S_{n-1} \oplus S_{n-1}$ is generated by*

$$\{v^1_{(i,M)}\}_{(i,M)\in\widetilde{\Lambda^1}} \cup \{[AC^{n-2}]\,, [BC^{n-2}]\,, [C^{n-1}]\}\,.$$

Proof. Let G be the sum of the R-submodule of F^1_2 generated by the elements stated above and $\mathfrak{m}F^1_2$. It is enough to show that $[\mathrm{m}^{n-1}_{A,B,C}]$ and $\langle\mathrm{m}^{n-1}_{A,B,C}\rangle$ are contained in G.

First, let us prove $\langle L\rangle \in G$ for any $L \in \mathrm{m}^{n-1}_{A,B,C}$. We write $L = A^\alpha B^\beta C^\gamma$, where $0 \le \alpha, \beta, \gamma \in \mathbb{Z}$ and $\alpha + \beta + \gamma = n - 1$. If $\alpha > 0$,

$$\langle L\rangle = \langle A\cdot A^{\alpha-1}B^\beta C^\gamma\rangle = v^1_{(1,A^{\alpha-1}B^\beta C^\gamma)} + x\cdot[C\cdot A^{\alpha-1}B^\beta C^\gamma] \in G\,.$$

Hence, we have to consider the case where $\alpha = 0$. However, as

$$\langle C^{n-1}\rangle = \langle C\cdot C^{n-2}\rangle = v^1_{(3,C^{n-2})} + [B\cdot C^{n-2}] \in G \quad \text{and}$$
$$\langle BC^{n-2}\rangle = \langle B\cdot C^{n-2}\rangle = v^1_{(2,C^{n-2})} + [A\cdot C^{n-2}] \in G\,,$$

we may assume $\beta \geq 2$. Then, as

$$\begin{aligned}\langle L\rangle &= \langle B\cdot B^{\beta-1}C^{\gamma}\rangle \\ &= v^1_{(2,B^{\beta-1}C^{\gamma})} + [A\cdot B^{\beta-1}C^{\gamma}] \\ &= v^1_{(2,B^{\beta-1}C^{\gamma})} + [B\cdot AB^{\beta-2}C^{\gamma}] \\ &= v^1_{(2,B^{\beta-1}C^{\gamma})} - v^1_{(3,AB^{\beta-2}C^{\gamma})} + \langle C\cdot AB^{\beta-2}C^{\gamma}\rangle\end{aligned}$$

and as $\langle C\cdot AB^{\beta-2}C^{\gamma}\rangle = \langle A\cdot B^{\beta-2}C^{\gamma+1}\rangle$, we get $\langle L\rangle \in G$.

Next, we prove $[\mathrm{m}^{n-1}_{A,B,C}] \subseteq G$. Let us notice $\mathrm{m}^{n-1}_{A,B,C} = A\cdot\mathrm{m}^{n-2}_{A,B,C} \cup B\cdot\mathrm{m}^{n-2}_{B,C} \cup \{C^{n-1}\}$. For any $M \in \mathrm{m}^{n-2}_{A,B,C}$ and any $X \in \mathrm{m}^{n-2}_{B,C}$, we have

$$[AM] = -v^1_{(2,M)} + \langle BM\rangle \in G \quad \text{and} \quad [BX] = -v^1_{(3,X)} + \langle CX\rangle \in G\,.$$

Hence, the proof is complete as $[C^{n-1}] \in G$ holds obviously.

Now, let q be the largest integer such that $q \leq n/2$. For any $1 \leq k \leq q$, we would like to construct an acyclic complex

$$0 \longrightarrow F^k_3 \xrightarrow{\varphi^k_3} F^k_2 \xrightarrow{\varphi^k_2} F^k_1 \xrightarrow{\varphi^k_1} F^k_0 = R$$

of finitely generated free R-modules satisfying the following conditions.

($\sharp^k_1$) $\mathrm{Im}\,\varphi^k_1 = I^n : \mathfrak{m}^{k-1}$.

($\sharp^k_2$) F^k_3 has an R-free basis indexed by $\Lambda^k := \mathrm{m}^{n-2k}_{A,B,C}$, say $\{w^k_M\}_{M\in\Lambda^k}$.

($\sharp^k_3$) Let $\widetilde{\Lambda^k} = \{1,2,3\} \times \Lambda^k$. Then, there exists a family $\{v^k_{(i,M)}\}_{(i,M)\in\widetilde{\Lambda^k}}$ of elements in F^k_2 satisfying the following conditions.

(i) For any $M \in \Lambda^k$, $\varphi^k_3(w^k_{(i,M)}) = x\cdot v^k_{(1,M)} + y\cdot v^k_{(2,M)} + z\cdot v^k_{(3,M)}$.

(ii) If $k < q$, for any $N \in \Lambda^{k+1} := \mathrm{m}^{n-2k-2}_{A,B,C}$,

$$v^k_{(3,A^2N)} = v^k_{(2,ABN)} - v^k_{(1,B^2N)} + v^k_{(1,ACN)} - x\cdot v^k_{(2,C^2N)} + x\cdot v^k_{(3,BCN)}\,.$$

(iii) There exists a subset U^k of F^k_2 such that $\{v^k_{(i,M)}\}_{(i,M)\in\widetilde{\Lambda^k}} \cup U^k$ generates F^k_2 and

$$|U^k| = \mathrm{rank}\,F^k_2 - 3\cdot\binom{n-2k+2}{2} + \binom{n-2k}{2}\,,$$

where the last binomial coefficient is regarded as 0 if $k = q$.

Let us notice that the acyclic complex $(F^1_\bullet, \varphi^1_\bullet)$, which is already constructed, satisfies $(\sharp^1_1)$, $(\sharp^1_2)$ (w^1_M is M itself for $M \in \Lambda^1$) and $(\sharp^1_3)$. So, we assume $1 \le k < q$ and an acyclic complex $(F^k_\bullet, \varphi^k_\bullet)$ satisfying the required conditions is given. Taking the $*$-transform of $(F^k_\bullet, \varphi^k_\bullet)$ with respect to x, y, z, we would like to construct $(F^{k+1}_\bullet, \varphi^{k+1}_\bullet)$.

First, we have the following result since the conditions $(\sharp^k_1)$, $(\sharp^k_2)$ and (i) of $(\sharp^k_3)$ are satisfied and $(I^n :_R \mathfrak{m}^{k-1}) :_R \mathfrak{m} = I^n :_R \mathfrak{m}^k$.

Lemma 5.4 *$(I^n :_R \mathfrak{m}^k)/(I^n :_R \mathfrak{m}^{k-1}) \cong F^k_3/\mathfrak{m}F^k_3$, so*

$$\ell_R(\,(I^n :_R \mathfrak{m}^k)/(I^n :_R \mathfrak{m}^{k-1})\,) = \binom{n-2k+2}{2}.$$

If Γ is a subset of Λ^k and $1 \le i \le 3$, we denote by (i, Γ) the subset $\{(i, M) \mid M \in \Gamma\}$ of $\widetilde{\Lambda^k}$. Let us notice that Λ^k is a disjoint union of $A^2{\cdot}\Lambda^{k+1}$, $A{\cdot}\mathrm{m}^{n-2k-1}_{B,C}$ and $\mathrm{m}^{n-2k}_{B,C}$. We set

$$'\Lambda^k = (1, \Lambda^k) \cup (2, \Lambda^k) \cup (3, A{\cdot}\mathrm{m}^{n-2k-1}_{B,C} \cup \mathrm{m}^{n-2k}_{B,C}).$$

Then the next result holds.

Lemma 5.5 *$\{v^k_{(i,M)}\}_{(i,M)\in '\Lambda^k} \cup U^k$ is an R-free basis of F^k_2.*

Proof. Because

$$\begin{aligned} |'\Lambda^k| &= 2 \cdot |\Lambda^k| + |A{\cdot}\mathrm{m}^{n-2k-1}_{B,C} \cup \mathrm{m}^{n-2k}_{B,C}| \\ &= 2 \cdot |\Lambda^k| + |\Lambda^k \setminus A^2{\cdot}\Lambda^{k+1}| \\ &= 3 \cdot |\Lambda^k| - |\Lambda^{k+1}| \\ &= 3 \cdot \binom{n-2k+2}{2} - \binom{n-2k}{2}, \end{aligned}$$

by (iii) of $(\sharp^k_3)$ we have $|'\Lambda^k| + |U^k| = \operatorname{rank} F^k_2$. Hence, by 3.3 it is enough to show that, for any $N \in \Lambda^{k+1}$, $v^k_{(3,A^2N)}$ is contained in the sum of the R-submodule of F^k_3 generated by $\{v^k_{(i,M)}\}_{(i,M)\in '\Lambda^k} \cup U^k$ and $\mathfrak{m}F^k_2$. We write $N = A^\alpha X$, where $X \in \mathrm{m}^{n-2k-2-\alpha}_{B,C}$. Then, using the equalities in (ii) of $(\sharp^k_3)$, the required containment can be proved by induction on α.

Let ${}^*F^k_2$ be the R-submodule of ${}'F^k_2$ generated by

$$\{[i, M]\}_{(i,M)\in\widetilde{\Lambda^k}} \cup \langle U^k \rangle,$$

where $[i, M] = [e_i \otimes w^k_M]$ for any $(i, M) \in \widetilde{\Lambda^k}$, and let ${}^*\varphi^k_2$ be the restriction of ${}'\varphi^k_2$ to ${}^*F^k_2$. In order to define ${}^*F^k_3$, we notice $\widetilde{\Lambda^k} \setminus {}'\Lambda^k = \{(3, A^2N) \mid N \in \Lambda^{k+1}\}$. Looking at (ii) of $(\sharp^k_3)$, we define ${}^*w^k_{(3,A^2N)} \in {}'F^k_3$ to be

$$-\check{e}_3 \otimes w^k_{A^2N} - \check{e}_2 \otimes w^k_{ABN} - \check{e}_1 \otimes w^k_{B^2N} + \check{e}_1 \otimes w^k_{ACN} + x \cdot \check{e}_2 \otimes w^k_{C^2N} + x \cdot \check{e}_3 \otimes w^k_{BCN}$$

for any $N \in \Lambda^{k+1}$. Let ${}^*F_3^k$ be the R-submodule of ${}'F_3^k$ generated by $\{{}^*w^k_{(3,A^2N)}\}_{N \in \Lambda^{k+1}}$ and let ${}^*\varphi_3^k$ be the restriction of ${}'\varphi_3^k$ to ${}^*F_3^k$. Thus we get a complex

$$0 \longrightarrow {}^*F_3^k \xrightarrow{{}^*\varphi_3^k} {}^*F_2^k \xrightarrow{{}^*\varphi_2^k} {}^*F_1^k \xrightarrow{{}^*\varphi_1^k} {}^*F_0^k = R\,.$$

Let us denote $({}^*F_\bullet^k, {}^*\varphi_\bullet^k)$ by $(F_\bullet^{k+1}, \varphi_\bullet^{k+1})$. Moreover, we put $w_N^{k+1} = {}^*w^k_{(3,A^2N)}$ for any $N \in \Lambda^{k+1}$. Then, by 3.5 and 3.6 we have the next result.

Lemma 5.6 $(F_\bullet^{k+1}, \varphi_\bullet^{k+1})$ *is an acyclic complex satisfying the following conditions.*

(1) $\operatorname{Im}\varphi_1^{k+1} = I^n :_R \mathfrak{m}^k$.

(2) $\{w_N^{k+1}\}_{N \in \Lambda^{k+1}}$ *is an R-free basis of* F_3^{k+1}.

(3) $\{[i,M]\}_{(i,M) \in \widetilde{\Lambda^k}} \cup \langle U^k \rangle$ *is an R-free basis of* F_2^{k+1}.

(4) *For any $N \in \Lambda^{k+1}$, the following equality holds ;*

$$\begin{aligned}\varphi_3^{K+1}(w_N^{k+1}) = &-x\cdot[2,A^2N] + y\cdot[1,A^2N] - x\cdot[3,ABN] + z\cdot[1,ABN]\\ &- y\cdot[3,B^2N] + z\cdot[2,B^2N] + y\cdot[3,ACN] - z\cdot[2,ACN]\\ &+ x^2[3,C^2N] - xz\cdot[1,C^2N] + x^2[2,BCN] - xy\cdot[1,BCN]\,.\end{aligned}$$

The assertions (1) and (2) of the lemma above imply that $(F_\bullet^{k+1}, \varphi_\bullet^{k+1})$ satisfies $(\sharp_1^{k+1})$ and $(\sharp_2^{k+1})$, respectively. Moreover, by (4) we have

$$\varphi_3^{k+1}(w_N^{k+1}) = x\cdot v^{k+1}_{(1,N)} + y\cdot v^{k+1}_{(2,N)} + z\cdot v^{k+1}_{(3,N)}$$

for any $N \in \Lambda^{k+1}$, where

$$\begin{aligned}v^{k+1}_{(1,N)} &= -[2,A^2N] - [3,ABN] + x\cdot[3,C^2N] + x\cdot[2,BCN]\,,\\ v^{k+1}_{(2,N)} &= [1,A^2N] - [3,B^2N] + [3,ACN] - x\cdot[1,BCN]\,,\\ v^{k+1}_{(3,N)} &= [1,ABN] + [2,B^2N] - [2,ACN] - x\cdot[1,C^2N]\,.\end{aligned}$$

Thus a family $\{v^{k+1}_{(i,N)}\}_{(i,N) \in \widetilde{\Lambda^{k+1}}}$ of elements in F_2^{k+1} satisfying (i) of $(\sharp_3^{k+1})$ is fixed, where $\widetilde{\Lambda^{k+1}} = \{1,2,3\} \times \Lambda^{k+1}$. The next result, which can be checked directly, insists that (ii) of $(\sharp_3^{k+1})$ is satisfied if $k+1 < q$.

Lemma 5.7 *Suppose $k+1<q$. Then $n-2k-4 \geq 0$ and we have*

$$v^{k+1}_{(3,A^2L)} = v^{k+1}_{(2,ABL)} - v^{k+1}_{(1,B^2L)} + v^{k+1}_{(1,ACL)} - x\cdot v^{k+1}_{(2,C^2L)} + x\cdot v^{k+1}_{(3,BCL)}$$

for any $L \in \Lambda^{k+2} := \mathrm{m}^{n-2k-4}_{A,B,C}$.

If Γ is a subset of Λ^k and $1 \leq i \leq 3$, we denote by $[\,i,\Gamma\,]$ the family $\{[i,M]\}_{M \in \Gamma}$ of elements in $K_1 \otimes_R F_3^k$. The next result means that (iii) of $(\sharp_3^{k+1})$ is satisfied.

Lemma 5.8 *We set*

$$U^{k+1} = [1, A\cdot \mathrm{m}_{B,C}^{n-2k-1} \cup \mathrm{m}_{B,C}^{n-2k}] \cup [\,3, \Lambda^k\,] \cup \\ \{[2, ABC^{n-2k-2}]\,, [2, AC^{n-2k-1}]\,, [2, BC^{n-2k-1}]\,, [2, C^{n-2k}]\} \cup \langle U^k \rangle\,.$$

Then $\{v_{(i,N)}^{k+1}\}_{(i,N)\,\in\,\widetilde{\Lambda^{k+1}}} \cup U^{k+1}$ *generates* F_2^{k+1} *and*

$$|U^{k+1}| = \operatorname{rank} F_2^{k+1} - 3\cdot\binom{n-2k}{2} + \binom{n-2k-2}{2}.$$

Proof. Let G be the sum of the R-submodule of F_2^{k+1} generated by $\{v_{(i,N)}^{k+1}\}_{(i,N)\,\in\,\widetilde{\Lambda^{k+1}}} \cup U^{k+1}$ and $\mathfrak{m}F_2^{k+1}$. We would like to show $G = F_2^{k+1}$. Let us recall that

$$[\,1, \Lambda^k\,] \cup [\,2, \Lambda^k\,] \cup [\,3, \Lambda^k\,] \cup \langle U^k \rangle$$

is an R-free basis of F_2^{k+1} and notice that Λ^k is a disjoint union of $A^2\cdot\Lambda^{k+1}$, $A\cdot\mathrm{m}_{A,B,C}^{n-2k-1}$ and $\mathrm{m}_{B,C}^{n-2k}$. Because $[3, \Lambda^k] \subseteq U^{k+1}$, it is enough to show $[\,1, A^2\cdot\Lambda^{k+1}\,] \cup [\,2, \Lambda^k\,] \subseteq G$.

First, we prove $[\,1, A^2\cdot\Lambda^{k+1}\,] \subseteq G$. Let us take any $N \in \Lambda^{k+1}$. Then

$$[1, A^2N] = v_{(2,N)}^{k+1} + [3, B^2N] - [3, ACN] + x\cdot[1, BCN] \in G\,,$$

and so the required inclusion follows.

Next, we prove $[\,2, \Lambda^k] \subseteq G$. Because

$$[2, A^2N] = -v_{(1,N)}^{k+1} - [3, ABN] + x\cdot[3, C^2N] + x\cdot[2, BCN] \in G$$

for any $N \in \Lambda^{k+1}$, we have $[2, A^2\cdot\Lambda^{k+1}] \subseteq G$. Furthermore, for any $B^\beta C^\gamma \in \mathrm{m}_{B,C}^{n-2k-1}$, we get $[2, AB^\beta C^\gamma] \in G$. In fact, $[2, AB^\beta C^\gamma] \in U^{k+1}$ if $\beta = 0$ or 1, and if $\beta \geq 2$, we have

$$\begin{aligned}[2, AB^\beta C^\gamma] &= [2, B^2\cdot AB^{\beta-2}C^\gamma] \\ &= v_{(3,AB^{\beta-2}C^\gamma)}^{k+1} - [1, A^2B^{\beta-1}C^\gamma] + [2, A^2B^{\beta-2}C^{\gamma+1}] + \\ &\qquad\qquad x\cdot[1, AB^{\beta-2}C^{\gamma+2}] \in G\,.\end{aligned}$$

Hence $[2, A\cdot\mathrm{m}_{B,C}^{n-2k-1}] \subseteq G$. In order to prove $[2, \mathrm{m}_{B,C}^{n-2k}] \subseteq G$, we newly take any $B^\beta C^\gamma \in \mathrm{m}_{B,C}^{n-2k}$. We have $[2, B^\beta C^\gamma] \in U^{k+1}$ if $\beta = 0$ or 1, and if $\beta \geq 2$, we have

$$\begin{aligned}[2, B^\beta C^\gamma] &= [2, B^2\cdot B^{\beta-2}C^\gamma] \\ &= v_{(3,B^{\beta-2}C^\gamma)}^{k+1} - [1, AB^{\beta-1}C^\gamma] + [2, AB^{\beta-2}C^{\gamma+1}] + x\cdot[1, B^{\beta-2}C^{\gamma+2}] \in G\,.\end{aligned}$$

Hence the required inclusion follows, and we have seen the first assertion of the theorem.

By (3) of 5.6 we have $\operatorname{rank} F_2^{k+1} = 3\cdot|\Lambda^k| + |U^k|$. On the other hand,

$$\begin{aligned}|U^{k+1}| &= |\Lambda^k \setminus A^2\cdot\Lambda^{k+1}| + |\Lambda^k| + 4 + |U^k| \\ &= 2\cdot|\Lambda^k| - |\Lambda^{k+1}| + 4 + |U^k|\,.\end{aligned}$$

Hence we get

$$\begin{aligned} \operatorname{rank} F_2^{k+1} - |U^{k+1}| &= |\Lambda^k| + |\Lambda^{k+1}| - 4 \\ &= \binom{n-2k+2}{2} + \binom{n-2k}{2} - 4 \\ &= 3\cdot\binom{n-2k}{2} - \binom{n-2k-2}{2}, \end{aligned}$$

and so the second assertion holds.

Thus we have constructed an acyclic complex

$$0 \longrightarrow F_3^q \xrightarrow{\varphi_3^q} F_2^q \xrightarrow{\varphi_2^q} F_1^q \xrightarrow{\varphi_1^q} F_0^q = R$$

of finitely generated free R-modules satisfying $(\sharp_1^q)$, $(\sharp_2^q)$ and $(\sharp_3^q)$. Of course, $n-2q = 0$ or 1, and

$$\Lambda^q = \begin{cases} \{1\} & \text{if } n-2q = 0 \\ \{A, B, C\} & \text{if } n-2q = 1\,. \end{cases}$$

The second condition of (iii) of $(\sharp_3^q)$ implies $\operatorname{rank} F_2^q = |\widetilde{\Lambda^q}| + |U^q|$. Hence, by the first condition of (iii) of $(\sharp_3^q)$, we see that $\{v_{(i,M)}^q\}_{(i,M)\in\widetilde{\Lambda^q}} \cup \langle U^q \rangle$ must be an R-free basis of F_2^q. Therefore, by 3.4 we get the next result.

Theorem 5.9 $\operatorname{depth} R/(I^n :_R \mathfrak{m}^q) > 0$, *and so* $I^{(n)} = I^n :_R \mathfrak{m}^q$.

Let us compute $\ell_R(I^{(n)}/I^n)$. By 5.9 and 5.4 we have

$$\ell_R(I^{(n)}/I^n) = \sum_{k=1}^{q} \ell_R((I^n : \mathfrak{m}^k)/(I^n : \mathfrak{m}^{k-1})) = \sum_{k=1}^{q} \binom{n-2k+2}{2}.$$

As a consequence, we get the next result.

Theorem 5.10 *The following equality holds ;*

$$\ell_R(I^{(n)}/I^n) = \begin{cases} \dfrac{1}{2}\dbinom{n+2}{3} - \dfrac{1}{4}\dbinom{n+1}{2} - \dfrac{1}{8}\dbinom{n}{1} - \dfrac{1}{8} & \textit{if } n \textit{ is even,} \\[2ex] \dfrac{1}{2}\dbinom{n+2}{3} - \dfrac{1}{4}\dbinom{n+1}{2} - \dfrac{1}{8}\dbinom{n}{1} & \textit{if } n \textit{ is odd.} \end{cases}$$

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